HARMONIC MORPHISMS AND DEFORMATION OF MINIMAL SURFACES IN MANIFOLDS OF DIMENSION 4

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MORPHISMES HARMONIQUES ET DEFORMATION DE SURFACES MINIMALES DANS DES VARIÉTÉS DE DIMENSION 4

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Abstract

In this thesis, we are interested in harmonic morphisms between Riemannian manifolds (M^m, g) and (N^n, h) for $m > n$. Such a smooth map is a harmonic morphism if it pulls back local harmonic functions to local harmonic functions: if $f : V \longrightarrow \mathbb{R}$ is a harmonic function on an open subset V on N and $\phi^{-1}(V)$ is non-empty, then the composition $f \circ \phi : \phi^{-1}(V) \longrightarrow \mathbb{R}$ is harmonic. The conformal transformations of the complex plane are harmonic morphisms.

In the late 1970's Fuglede and Ishihara published two papers ([Fu]) and ([Is]), where they discuss their results on *harmonic morphisms* or *mappings preserving harmonic functions*. They characterize non-constant harmonic morphisms $F : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds as those harmonic maps, which are horizontally conformal, where F horizontally conformal means : for any $x \in M$ with $dF(x) \neq 0$, the restriction of $dF(x)$ to the orthogonal complement of $kerdF(x)$ in T_xM is conformal and surjective. This means that we are dealing with a special class of harmonic maps.

The theory becomes very successful when the codomain N is a surface. In this case, the harmonic morphisms have particular properties. Note the conformal invariance : the equations of harmonic morphisms with values in a Riemannien manifold N of dimension 2 depend only of the conformal class of the metric on N.

In this thesis we focus on the case when the codomain is a surface. The geometric characterization due to Baird and Eells ([B-E]) reduces the problem of harmonicity to the minimality of the fibres (at regular points): If $dimN = 2$ a horizontally conformal map is harmonic (then a harmonic morphism) if and only if the regular fibers are minimal in M. If $dimN > 2$, a harmonic morphism has minimal fibers if and only if it is horizontally homothetic.

The study of harmonic morphisms from 4-dimensional Einstein manifolds to Riemann surfaces has been greatly simplified with the aid of twistor methods. In some cases even, a complete classification of these maps has been found, see [B-W, Vi1, Wo].

J. Wood in [Wo] (complemented by M. Ville [Vi1]) proved that when the domain is a 4-dimensional Einstein manifold and the codomain is a surface, a harmonic morphism can be equivalently characterized as a map which is holomorphic with respect to some integrable Hermitian structure on its domain and has superminimal fibres. It follows that there do not exist non-constant harmonic morphisms from $(\mathbb{S}^4, can)$ into a Riemann surface.

In chapter 2, we investigate the structure of a harmonic morphism F from a Riemannian 4-manifold M^4 to a 2-surface N^2 near a critical point m_0. If m_0 is an isolated critical point or if M^4 is compact without boundary, we show that F is pseudo-holomorphic w.r.t. an almost Hermitian structure defined in a neighbourhood of m_0. If M^4 is compact without boundary, the singular fibres of F are branched minimal surfaces.

In chapter 3, we study examples of harmonic morphisms due to Burel from $(\mathbb{S}^4, g_{k,l})$ into $\mathbb{S}^2$ where $(g_{k,l})$ is a family of metrics which are conformal to the canonical metric. To do this construction we define the two maps, F from $(\mathbb{S}^4, g_{k,l})$ to $(\mathbb{S}^3, \bar{g_{k,l}})$ and $\varphi_{k,l}$ from $(\mathbb{S}^3, \bar{g_{k,l}})$ to $(\mathbb{S}^2, can)$; these two maps are both horizontally conformal and harmonic. The map $\Phi_{k,l} = \varphi_{k,l} \circ F$ is a harmonic morphism. It follows from Baird-Eells that the regular fibres of $\Phi_{k,l}$ for every k, l are minimal. If $|k| = |l| = 1$, the set of critical points is given by the preimage of the north pole : it consists in two 2-spheres meeting transversally at 2 points. If $k, l \neq 1$ the set of critical points are the preimages of the north pole (the same two spheres as for $k = l = 1$ but with multiplicity l) together with the preimage of the south pole (a torus) with multiplicity k.

Finally, in chapter 4, we investigate a construction by Baird-Ou of harmonic morphisms from open sets of $(\mathbb{S}^2 \times \mathbb{S}^2, can)$ to a 2-surface $\mathbb{S}^2$. We check that they are holomorphic with respect to one of the four canonical Hermitian complex structures.

Keywords : Harmonic Morphism, Minimal Surface, Almost Complex Structure.

Contents

Chapter 1

Introduction

1.1 Preliminaries

Here we remind the reader of some important notions around harmonic morphisms that we needed. We used the book [B-W] and we refer the reader to it for more details.

Given a smooth map $\phi : M \longrightarrow N$ between manifolds, its differential will be denoted by $d\phi : TM \longrightarrow TN$; for each $x \in M$, this restricts to a linear mapping $d\phi_x : T_xM \longrightarrow T_{\phi(x)}N$ called the differential at x. In local coordinates $(x^1, ..., x^m)$ on M and $(y^1, ..., y^n)$ on N, we shall write $\phi^\alpha = y^\alpha \circ \phi$ and $\phi_i^\alpha = \partial\phi^\alpha/\partial x^i$,so that

$$d\phi(\partial/\partial x^i) = \sum_{\alpha=1}^{n} \phi_i^\alpha \partial/\partial y^\alpha \tag{1.1}$$

More generally, given bases $\{X_i\}$ and $\{Y_\alpha\}$ of T_xM and $T_{\phi(x)}N$, respectively, we write

$$d\phi_x(X_i) = \sum_{\alpha=1}^{n} \phi_i^\alpha Y_\alpha. \tag{1.2}$$

If $N = \mathbb{R}$, we can identify the differential $d\phi$ with a 1-form on M, which we also denote by $d\phi$.
By a (smooth) Riemannian metric g on a (smooth) manifold M we mean a symmetric positive-definite inner product on each tangent space:

$$g_x : T_xM \times T_xM \longrightarrow \mathbb{R}, \quad (v, w) \longmapsto g(v, w) \quad (x \in M,\ v, w \in T_xM) \tag{1.3}$$

which varies smoothly with x (i.e., for any smooth vector fields E and F, the function $x \longmapsto g(E_x, F_x)$ is smooth). We shall often use the notation

$$\langle v, w\rangle = g(v, w) \tag{1.4}$$

when the metric is clear from the context. Canonical or standard metrics will often denoted by *can*. The corresponding norm will be denoted by $|v| = \sqrt{\langle v, v\rangle}$, so that the square norm is given by $|v|^2 = \langle v, v\rangle$. Later on, we shall find it convenient to extend the complexified tangent bundle given by $T^cM = TM \otimes_{\mathbb{R}} \mathbb{C}$.

For the rest of this section, (M,g) will be a smooth Riemannian manifold.
In local coordinates $(x^1, ..., x^m)$ on M we write $g = \sum_{i,j} g_{ij} dx^i dx^j$; thus we have $g(\partial x^i, \partial x^j) = g_{ij}$. More generally, if $\{X_i\}$ is a frame on M, we write $g_{ij} = g(X_i, X_j)$. The *gradient* of a smooth function $f : M \longrightarrow \mathbb{R}$ is the vector field characterized by

$$g(gradf, E) = df(E) \quad (x \in M, E \in T_xM) \tag{1.5}$$

In local coordinates, it has the expression

$$gradf = \sum_{i,j} \left(g^{ij} \frac{\partial f}{\partial x^j} \right) \frac{\partial}{\partial x^i}. \tag{1.6}$$

where (g^{ij}) is the inverse matrix of (g_{ij}). We let $\nabla = \nabla^M$ denote the Levi-Civita connection on M determined by the formula: for any vector fields E,F and G we have

$$\begin{aligned} \nabla_E F - \nabla_F E &= [E, F]; \\ E(g(F, G)) &= g(\nabla_E F, G) + g(F, \nabla_E G). \end{aligned} \tag{1.7}$$

The Levi-Civita connection induces connections on other tensor bundles, e.g., if θ is a 1-form and E a vector field, $\nabla_E \theta$ is the 1-form given by

$$(\nabla_E \theta)(G) = E(\theta(G)) - \theta(\nabla_E G) \quad (G \in \Gamma(TM)),$$

i.e. we have the Leibniz product rule

$$E(\theta(G)) = (\nabla_E \theta)(G) + \theta(\nabla_E G) \quad (G \in \Gamma(TM).$$

Equation (1.7) can then written in the form

$$\nabla g = 0.$$

Now let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds and let $x \in M$. The Hilbert-Schmidt norm $|d\phi_x|$ of its differential at x is defined by

$$|d\phi_x|^2 = \sum_{i=1}^{m} h(d\phi_x(e_i), d\phi_x(e_i)), \tag{1.8}$$

where $\{e_i\}$ is an orthonormal basis for T_xM.
Define the pull-back $\phi^* h$ of the metric h by

$$\phi^* h(E, F) = h(d\phi_x(E), d\phi_x(F)) \quad (E, F \in T_xM); \tag{1.9}$$

then we have

$$|d\phi_x|^2 = Tr_g \phi^* h = \sum_{i=1}^{m} \phi^* h(e_i, e_i). \tag{1.10}$$

1.2 Harmonic maps and minimal surfaces

Definition 1. *Let $\phi : M \longrightarrow N$ be a smooth mapping between Riemannian manifolds. Then ϕ is called a harmonic morphism if, for every harmonic function $f : V \longrightarrow \mathbb{R}$ defined on an open subset V of N with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$*

Thus a harmonic morphism is a smooth map which pulls back (local) harmonic functions to harmonic functions; equivalently, it pulls back germs of harmonic functions to germs of harmonic functions. The most obvious examples of harmonic morphisms are constant maps and isometries.

1.2.1 Horizontally weakly conformal maps

For any smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ between Riemannian manifolds, and any regular point $x \in M$, set $V_x = \text{Ker } d\phi_x$ and $H_x = V_x^{\perp}$; then V_x is called the vertical space and H_x the horizontal space of ϕ at x.

Definition 2. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then ϕ is called horizontally weakly conformal or semiconformal at x if either*
(i) $d\phi_x = 0$,or
(ii) $d\phi_x$ maps the horizontal space $H_x = \{Ker(d\phi_x)\}^{\perp}$ conformally onto $T_{\phi(x)}N$, i.e., $d\phi_x$ is surjective and there exists a dilation factor $\lambda(x) \neq 0$
such that

$$h(d\phi_x(X), d\phi_x(Y)) = \lambda^2(x)g(X, Y) \quad (X, Y \in H_x).$$

Since a horizontally weakly conformal map is a submersion at regular points, we have the following restriction on dimensions.

Proposition 1. *Let $\phi : M \longrightarrow N$ be a horizontally weakly conformal map. If $dimM < dimN$, then ϕ is constant.*

1.2.2 Harmonic mappings

1.2.2.1 The Energy

Let (M^m, g) and (N^n, h) be Riemannian manifolds and let $\phi : M \longrightarrow N$ be a smooth mapping between them. The energy density of ϕ is the smooth function $e(\phi) : M \longrightarrow [0, \infty)$ given by

$$e(\phi)_x = \frac{1}{2}|d\phi_x|^2 \quad (x \in M), \tag{1.11}$$

where $|d\phi_x|$ denotes the Hilbert-Schmidt norm of $d\phi_x$ defined by (1.8).
Let D be a compact domain of M. The energy of ϕ over D is the integral of its energy

density:

$$E(\phi; D) = \int_D e(\phi) v_g = \frac{1}{2} \int_D |d\phi|^2 v_g, \tag{1.12}$$

where v_g denotes the volume form.
Note that $E(\phi; D) \geqslant 0$, with equality if and only if ϕ is constant on D. If M is compact, we write $E(\phi)$ for $E(\phi; M)$.
Let $C^\infty(M, N)$ denote the space of all smooth maps from M to N. A map $\phi : M \longrightarrow N$ is said to be harmonic if it is a critical point of the energy functional $E(.; D) : C^\infty(M, N) \longrightarrow \mathbb{R}$ for any compact domain D. (If M is compact it suffices to check this for D=M). We now explain this more fully. By a smooth variation of ϕ we mean a smooth map

$$\phi : M \times (-\epsilon, \epsilon) \longrightarrow N, (x, t) \longmapsto \phi_t(x)$$

$(\epsilon > 0)$ such that $\phi_0 = \phi$.

Definition 3. *A smooth map $\phi : (M, g) \longrightarrow (N, h)$ is said to be harmonic if*

$$\frac{d}{dt} E(\phi_t; D)|_{t=0} = 0 \tag{1.13}$$

for all compact domains D and all smooth variations $\{\phi_t\}$ of ϕ supported in D.

In order to understand this definition, we will now calculate the left-hand side of (1.13).

1.2.2.2 Tension field

Let $M = (M, g)$ and $N = (N, h)$ be Riemannian manifolds, and suppose that $\phi : M \longrightarrow N$ is a smooth mapping between them. The differential dϕ of ϕ can be viewed as a section of the bundle $T^*M \otimes \phi^{-1}TN = Hom(TM, \phi^{-1}TN) \longrightarrow M$. This bundle has a connection ∇ induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^ϕ of the Levi-Civita connection on N. On applying that connection to $d\phi$, we obtain the second fundamental form of ϕ:

$$B = \nabla d\phi \in \Gamma(T^*M \bigcirc T^*M \otimes \phi^{-1}TN). \tag{1.14}$$

Explicitly, for $X, Y \in \Gamma(TM)$,

$$\nabla d\phi(X, Y) = \nabla^\phi_X (d\phi(Y)) - d\phi(\nabla^M_X Y). \tag{1.15}$$

Now let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. On taking the trace of the second fundamental form, we obtain the following very important quantity.

Definition 4. *The tension field of ϕ is the section $\tau(\phi) \in \Gamma(\phi^{-1}TN)$ defined by*

$$\tau(\phi) = TrB = B(\phi)(e_1, e_2) = \nabla_{e_1} d\phi(e_1) + \nabla_{e_2} d\phi(e_2) \tag{1.16}$$

where $\{e_1, e_2\}$ is a normal frame on M.

Proposition 2. *(First variation of the energy) Let $\phi : M \longrightarrow N$ be a smooth map and let $\{\phi_t\}$ be a smooth variation of ϕ supported in D. Then*

$$\frac{d}{dt}E(\phi_t; D)|_{t=0} = -\int_D \langle v, \tau(\phi)\rangle v_g \tag{1.17}$$

where $v(x) = (\partial\phi_t/\partial t)(x)|_{t=0}$ denotes the variation vector field of $\{\phi_t\}$. Here $\langle .,.\rangle$ denotes the pull-back metric on $\phi^{-1}TN$; explicitly, at any point x of D, $\langle v, \tau(\phi)\rangle = h_{\phi(x)}(v(x), \tau(\phi)_x)$.

Theorem 1. *(Harmonic equation, Eells and Sampson, [Ee-Sam]) Let $\phi : M \longrightarrow N$ be a smooth map. Then ϕ is harmonic if and only if*

$$\tau(\phi) = 0. \tag{1.18}$$

1.2.3 Minimal submanifolds

Let $\phi : M^m \longrightarrow N^n$ be an isometric immersion. The mean curvature μ^M of ϕ (or of the immersed submanifold $\phi(M)$ in N) is defined by

$$\mu^M = \frac{1}{m}TrB = \frac{1}{m}\sum_{i=1}^{m} B(e_i, e_i) \quad ((e_i)\ an\ orthonormal frame); \tag{1.19}$$

here B denotes the second fundamental form of the immersed submanifold.

Definition 5. *Let $\phi : M^m \longrightarrow N^n$ be an immersion; then ϕ is minimal if and only if*

$$\mu^M = 0.$$

Proposition 3. *Let $\phi : M \longrightarrow N$ be an isometric immersion. Then*

$$\tau(\phi) = TrB = (dimM)\mu^M \tag{1.20}$$

Hence, an isometric immersion is harmonic if and only if it is a minimal immersion.

Proof. Let ϕ be an isometric embedding of dimension m.
Since $d\phi$ is an isometry, we can identify an orthonormal frame $(e_i)_i$ of TM with $(d\phi(e_i))_i$ and

$$\tau(\phi) = \sum_i \nabla^N_{e_i} e_i - \nabla^M_{e_i} e_i$$

It follows that

$$\tau(\phi) = \sum_i \nabla d\phi(e_i, e_i) = \sum_i B(e_i, e_i)$$

where B is the second fondamental form.
By definition 5 , $\tau(\phi) = 0$ if and only if ϕ is minimal. □

Proposition 4. *A conformal immersion ϕ from a Riemannian manifold M of dimension 2 (or conformal surface) is harmonic if and only if its image is minimal.*

Proof. Let the metric $\tilde{g} = \mu g$, where μ is a positive function on M. If (e_i) is an orthonormal basis for g then $\left(\frac{e_i}{\sqrt{\mu}}\right)$ is an orthonormal basis for $\tilde{g}$. The Hilbert-Schmidt norm becomes

$$|d\phi|^2_{\tilde{g}} = \frac{1}{\mu}|d\phi|^2_g,$$

and the volume form

$$dv_{\tilde{g}} = \mu dv_g.$$

Then the energy of ϕ depends only on the conformal class on M. So ϕ is harmonic for all the metrics on M conformal to g. ☐

1.2.4 Minimal surfaces and branch points

We now consider maps that are not necessarily immersions. To that effect we recall some facts on branch points according to [G-O-R]. Let M be a 2-dimensional differential manifold, and N an n-dimensional differentiable manifold, $n \geqslant 2$. Let $\phi : M \longrightarrow N$ be a differential map. For a point $p \in M$, let (u_1, u_2) be local coordinate around p. In place of (u_1, u_2) we shall often use the complex parameter $z = u_1 + iu_2$.

Definition 6. *The map ϕ has a branch point at p if there exists an integer $m \geqslant 2$ and local coordinates u_1, u_2 at $p, x_1, ..., x_n$ at $\phi(p)$ such that the map ϕ expressed in these coordinates takes the form*

$$\begin{aligned} x_1 + ix_2 &= z^m + \vartheta(z) \\ x_k &= \chi_k(z), \qquad\qquad k = 3, ..., n, \end{aligned}$$

where

$$\begin{aligned} \vartheta(z), \chi_k(z) &= o(|z|^m), \\ \frac{\partial\vartheta}{\partial u_j}(z), \frac{\partial\chi_k}{\partial u_j}(z) &= o(|z|^{m-1}) \quad j = 1, 2 \end{aligned}$$

More precisely, p is then called a branch point of order $m - 1$.

Definition 7. *A map ϕ from an oriented surface M into a manifold N (of dimension $\geqslant 2$) is a branched immersion if and only if it is an immersion except at branch points i.e. where the differential $d\phi$ vanishes and locally it is given by Definition 6.*

Notice that the set of branch points is a discrete set.

Proposition 5. *(see for example [Gau]) A non-constant conformal harmonic map from a Riemann surface into a Riemannian manifold is a branched immersion.*

Moreover proposition 4 extends to branched immersions. Indeed let ϕ be a branched minimal immersion. We know that $\tau(\phi) = 0$ at regular points. If $d\phi(x) = 0$, then ϕ is a branch point and x is the limit point of regular points, in which case $\tau(\phi)_x = 0$ by continuity.

Hence ϕ is harmonic if and only if $\tau(\phi)$ is zero at regular points, and this holds if and only if ϕ is minimal at regular points.
Going back to definition 6, we see that, in a neighborhood of a branch point

$$\frac{\partial\phi}{\partial u_1} \wedge \frac{\partial\phi}{\partial u_2} = m^2|z|^{2(m-1)}e_1 \wedge e_2 + o(|z|^{2(m-1)}) \tag{1.21}$$

(where $e_i = \frac{\partial}{\partial x_i}, i = 1, \cdots, n$). Thus we can define an oriented 2-plane bundle $T\phi$ on M by setting for a point $z \in M$:
i) if z is a regular point of ϕ : $(T\phi)_z := d\phi(T_zM)$,
ii) if z is a branch point $(T\phi)_z$ is the oriented plane generated by $e_1 \wedge e_2$ as in equation (1.21).

We call it the image tangent bundle of ϕ and denote it $T\phi$.
At a singular point of ϕ, the branching order of this point is the index at this point of the homomorphism $d\phi$ of $T\Sigma$ in $T\phi$.

1.2.5 Harmonic morphisms

Proposition 6. *The second fundamental form of the composition of two maps $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ is given by*

$$\nabla d(\psi \circ \phi) = d\psi(\nabla d\phi) + \nabla d\psi(d\phi, d\phi). \tag{1.22}$$

Corollary 1. *(Composition law) The tension field of the composition of two maps $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ is given by*

$$\tau(\psi \circ \phi) = d\psi(\tau(\phi)) + Tr\nabla d\psi(d\phi, d\phi). \tag{1.23}$$

Here $Tr\nabla d\psi(d\phi, d\phi) = \sum_{i=1}^{m} \nabla d\psi(d\phi(e_i), d\phi(e_i))$, where $\{e_i\}$ is an orthonormal frame.

Lemma 1. *Let M=(M,g) and N=(N,h) be Riemannian manifolds. A harmonic horizontally weakly conformal map $\phi : M \longrightarrow N$ is a harmonic morphism.*

Proof. Let f be a harmonic function on an open subset of N. The composition law (Corollary 1) yields

$$\tau(f \circ \phi) = df(\tau(\phi)) + Tr\nabla df(d\phi, d\phi). \tag{1.24}$$

As ϕ is harmonic, the first term vanishes. Let (e_i) be orthonormal basis on M consisting of vertical and horizontal vectors. If e_i is a vertical vector then $d\phi(e_i) = 0$. For the horizontal vectors e_i, $(d\phi(e_i))$ is an orthogonal basis of TN consisting of vectors of the same norm. Then

$$\tau(f \circ \phi) = \lambda^2 \; Tr\nabla df = \lambda^2\tau(f). \tag{1.25}$$

It follows that, if f is harmonic, so is $f \circ \phi$. Hence ϕ is harmonic morphism. □

Theorem 2. *(Characterization; [Fu], [Is]) A smooth map $\phi : M \longrightarrow N$ between Rienmannian manifolds is a harmonic morphism if and only if ϕ is both harmonic and horizontally weakly conformal.*

1.2.6 The mean curvature of the fibres

We now relate the tension field of a horizontally conformal submersion to the mean curvature of its fibres and the horizontal gradient of its dilation.

Proposition 7. *(Fundamental equation) Let $\phi : M^m \longrightarrow N^n$ be a smooth horizontally conformal submersion between Riemannian manifolds of dimensions $m, n \geqslant 1$. Let $\lambda : M \longrightarrow (0, \infty)$ denote the dilation of ϕ. Then the tension field of ϕ is given by*

$$\tau(\phi) = -(n-2)d\phi(grad\ ln\lambda) - (m-n)d\phi(\mu^{\nu}) \tag{1.26}$$

We shall call (1.26) the fundamental equation (for the tension field of a horizontally conformal submersion).

Theorem 3. *(Baird and Eells 1981) Let $\phi : M^m \longrightarrow N^n$ be a smooth non-constant horizontally weakly conformal map between Riemannian manifolds of dimensions $m, n \geqslant 1$. Then ϕ is harmonic, and so a harmonic morphism, if and only if, at every regular point, the mean curvature vector field μ^{ν} of the fibres and the gradient of the dilation λ of ϕ are related by*

$$(n-2)\mathfrak{H}(grad\ ln\lambda) + (m-n)\mu^{\nu} = 0, \tag{1.27}$$

where $\mathfrak{H}$ is the horizontal component on the horizontal space.
In particular, if n=2, then ϕ is harmonic, and so a harmonic morphism, if and only if, at every regular point, the fibres of ϕ are minimal.

Chapter 2

On harmonic morphisms from 4-manifolds to Riemann surfaces and local almost Hermitian structures

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We investigate the structure of a harmonic morphism F from a Riemannian 4-manifold M^4 to a 2-surface N^2 near a critical point m_0. If m_0 is an isolated critical point or if M^4 is compact without boundary, we show that F is pseudo-holomorphic w.r.t. an almost Hermitian structure defined in a neighbourhood of m_0.
If M^4 is compact without boundary, the singular fibres of F are branched minimal surfaces.

2.1 Introduction

2.1.1 Background

A harmonic morphism $F : M \longrightarrow N$ between two Riemannian manifolds (M, g) and (N, h) is a map which pulls back local harmonic functions on N to local harmonic functions on M. Although harmonic morphisms can be traced back to Jacobi, their study in modern times was initiated by Fuglede and Ishihara who characterized them using the notion of horizontal weak conformality, or semiconformality:

Definition 8. *(see [B-W] p.46) Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds and let $x \in M$. Then F is called horizontally weakly conformal at x if either*
1) $dF_x = 0$
2) dF_x maps the space $Ker(dF_x)^\perp$ conformally onto $T_{F(x)}N$, i.e. there exists a number $\lambda(x)$ called the dilation of F at x such that

$$\forall X, Y \in Ker(dF_x)^\perp, h(dF_x(X), dF_x(X)) = \lambda^2(x) g(X, Y).$$

The space $Ker(dF_x)$ (resp. $Ker(dF_x)^\perp$) is called the vertical (resp. horizontal) space at x.

Fuglede and Ishihara proved independently

Theorem 4. *([Fu],[Is]) Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. The following two statements are equivalent:*
1) For every harmonic function $f : V \longrightarrow \mathbb{R}$ defined on an open set V of N, the function $f \circ F$ defined on the open set $F^{-1}(V)$ of M is harmonic.
2) The map F is harmonic and horizontally weakly conformal.
Such a map is called a harmonic morphism.

When the target is 2-dimensional, Baird and Eells proved

Theorem 5. *([B-E]) Let $F : (M^m, g) \longrightarrow (N^2, h)$ be a smooth nonconstant horizontally weakly conformal map between a Riemannian manifold (M^m, g) and a Riemannian 2-surface (N^2, h). Then F is harmonic (hence a harmonic morphism) if and only if the fibres of F at regular points are minimal submanifolds of M.*

It follows from Th.5 that holomorphic maps from a Kähler manifold to a Riemann surface are harmonic morphisms; this raises the question of the interaction between harmonic morphisms to surfaces and holomorphic maps. John Wood studied harmonic morphisms $F : M^4 \longrightarrow N^2$ from an Einstein 4-manifold M^4 to a Riemann surface N^2 and exhibited an *integrable* Hermitian structure J on the regular points of F w.r.t. which F is holomorphic ([Wo]). He extended J to some of the critical points of F and the M. Ville extended it to all critical points ([Vi1]).
By contrast, Burel constructed many harmonic morphisms from $\mathbb{S}^4$ to $\mathbb{S}^2$, for non-canonical metrics on $\mathbb{S}^4$ ([Bu]); he was building upon previous constructions on product of spheres by Baird and Ou ([B-O]). Yet it is well-known that $\mathbb{S}^4$ does not admit any global almost complex structure (see for example [St] p.217).

2.1.2 The results

In the present paper, we continue along the lines of [Wo] and [Vi1] and investigate the case of a harmonic morphism $F : M^4 \longrightarrow N^2$ from a general Riemannian 4-manifold M^4 to a 2-surface N^2. In [Wo] the integrability of J follows from the Einstein condition so we cannot expect to derive an integrable Hermitian structure in the general case. Could F be pseudo-holomorphic w.r.t. some *almost* Hermitian structure J on M^4? Burel's example on $\mathbb{S}^4$ tells us that we cannot in general expect J to be defined on all of M^4: the most we can expect is for F to be pseudo-holomorphic w.r.t. a local almost Hermitian structure. We feel that this should be true in general; however, we only are able to prove it in two cases:

Theorem 6. *Let (M^4, g) be a Riemannian 4-manifold, let (N^2, h) be a Riemannian 2-surface and let $F : M^4 \longrightarrow N^2$ be a harmonic morphism. Consider a critical point m_0 in M^4 and assume that one of the following assertions is true*
1) m_0 is an isolated critical point of F
OR
2) (M^4, g) is compact without boundary (and m_0 need not be isolated).
Then there exists an almost Hermitian structure J defined in a neighbourhood of m_0 w.r.t. which F is pseudo-holomorphic.

NB. The pseudo-holomorphicity of F means: if $m \in M^4$ and $X \in T_m M^4$,

$$dF(JX) = j \circ dF(X)$$

where j denotes the complex structure on N^2.

Th.6 enables us to use the work of [McD] and [M-W] on pseudo-holomorphic curves to study singularities: the local topology of a singularity of a fibre of F is the same as the local topology of a singular complex curve in $\mathbb{C}^2$.

We derive from the proof of Th. 6

Corollary 2. *Let $F : M^4 \longrightarrow N^2$ be a harmonic morphism from a compact Riemannian 4-manifold without boundary to a Riemann surface and let u_0 be a singular value of F. Then the preimage $F^{-1}(u_0)$ is a (possibly branched) minimal surface.*

If the manifold M^4 is Einstein, the Hermitian structure constructed by Wood is parallel on the fibres of the harmonic morphism and has a fixed orientation. In the general case, around regular points of F, there are two local almost Hermitian structures making F pseudo-holomorphic; they have opposite orientations and we denote them J_+ and J_-. We follow Wood's computation without assuming M^4 to be Einstein and get a a bound on the product of the $\|\nabla J_\pm\|$'s (we had hoped for a local bound on one of the $\|\nabla J_\pm\|$'s):

Proposition 8. *Let (M^4, g) be a Riemannian 4-manifold, let (N^2, h) be a Riemannian 2-surface and let $F : M^4 \longrightarrow N^2$ be a harmonic morphism. We denote by j the complex structure on N^2 compatible with the metric and orientation. For a regular point m of F, we let J_+ (resp. J_-) be the almost complex structure on M^4 such that*
i) J_+ and J_- preserve the metric g
ii) J_+ (resp. J_-) preserves (resp. reverses) the orientation on $T_m M^4$
iii) the map $dF : (T_m M^4, J_\pm) \longrightarrow (N^2, j)$ is complex-linear.
Let K be a compact subset of M^4: there exists a constant A such that, for every regular point m of M^4 in K and every unit vertical tangent vector T at m,

$$\|\nabla_T J_+\| \|\nabla_T J_-\| \leqslant A$$

where ∇ denotes the connection induced by the Levi-Civita connection on M^4.

2.1.3 Sketch of the paper

In §2.2, we recall that the lowest order term of the Taylor development at a critical point of F is a homogeneous holomorphic polynomial; we use it to control one of the two local pseudo-Hermitian structures for which F is pseudo-holomorphic at regular points close to m_0. We express this in the Main Lemma (§2.2.3) and the first case of Th.6 follows almost immediately (§2.3).
In §2.4 we prove second case of Th. 6 using the twistor constructions of Eells and Salamon ([Ee-Sal]): the twistor space $Z(M^4)$ is a 2-sphere bundle above M^4 endowed with an almost complex structure $\mathcal{J}$ and the regular fibres of F lift to $\mathcal{J}$-holomorphic curves

in $Z(M^4)$. The assumptions of the second case of Th. 6 enable us to prove that these curves have bounded area so we can use Gromov's compacity theorem: as we approach m_0, the lifts of the regular fibres of F in each of the two twistor spaces of M^4 converge to a $\mathcal{J}$-holomorphic curve. The Main Lemma 2.2 enables us to pick one of the two orientations so that the limit curve has no vertical component near m_0: near m_0, it is the lift of the fibre of F containing m_0. This is the key point in the proof of Th. 6 2).
In §2.5, we prove Prop 8 using an identity which Wood established to prove the superminimality of the fibres in the Einstein case.

For background and detailed information about harmonic morphisms, we refer the reader to [B-W].

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2.2 The main lemma

2.2.1 The almost complex structure at regular points

A REMARK ABOUT THE NOTATION. If m is a point in M^4, we denote by $|m|$ the distance of m to m_0. We introduce several constants, which we number C_1,...,C_{10},...; they all have the same goal which is to say that one quantity or another is a $\mathcal{O}(|m|)$, so the reader in a hurry can ignore the indices and think of a single constant C.

Let m be a regular point of F in M^4; as we mentioned above in Def.8, the tangent space of M^4 at a regular point m of F splits as follows:

$$T_m M^4 = V_m \oplus H_m \tag{2.1}$$

where the vertical space V_m is the space tangent at m to the fibre $F^{-1}(F(m))$ and the horizontal space H_m is the orthogonal complement of V_m in $T_m M^4$.

2.2.2 The symbol and its extension in a neighbourhood of a critical point

We use the notations of Th.6 and we let m_0 be a critical point of F. We denote by k, $k > 1$, the order of F at m_0; namely, if (x_i) is a coordinate system centered at m_0, m_0 being identified with $(0, ..., 0)$, we have
1) for every multi-index $I = \{i_1, ..., i_4\}$ with $|I| \leqslant k-1$, $|I| = \sum_{j=1}^{4} i_j \leqslant k-1$,

$$\frac{\partial^{|I|} F}{\partial^{i_1} x_1 ... \partial^{i_4} x_4}(0, ..., 0) = 0$$

2) there exists a multi-index $J = \{j_1, ..., j_4\}$ with $|J| = k$ such that

$$\frac{\partial^k F}{\partial^{j_1} x_1 ... \partial^{j_4} x_4}(0, ..., 0) \neq 0$$

The lowest order term of the Taylor development of F at m_0 is a homogeneous polynomial

$$P_0 : T_{m_0} M^4 \longrightarrow T_{F(m_0)} N^2$$

of degree k called the *symbol* of F at m_0. Fuglede showed ([Fu]) that P_0 is a harmonic morphism between $T_{m_0} M^4$ and $T_{F(m_0)} N^2$; it follows from [Wo] that P is a holomorphic polynomial of degree k for some orthogonal complex structure J_0 on $T_{m_0} M^4$.
REMARK. The complex structure J_0 is not always uniquely defined as the following two examples illustrate:
1) $P_0(z_1, z_2) = z_1 z_2$: J_0 is uniquely defined
2) $P_0(z_1, z_2) = z_1^2$: there are two possible J_0's with opposite orientations.

2.2.3 The main lemma

We identify a neighbourhood U of m_0 with a ball in $\mathbb{R}^4$, the point m_0 being identified with the origin and we let (x_i) be a system of normal coordinates in U. We pick these coordinates so that, at the point m_0, we have

$$J_0 \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} \qquad J_0 \frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4} \tag{2.2}$$

We extend J_0 in U by requiring (2.2) to be verified for all points in U. Of course J_0 does not necessarily preserve the metric outside of m_0, nevertheless there exists a constant C_2 such that, for a vector X tangent at a point m

$$| < J_0 X, J_0 X > - < X, X > | \leqslant C_2 |m|^2 \|X\|^2 \qquad | < J_0 X, X > | \leqslant C_2 |m|^2 \|X\|^2 \tag{2.3}$$

We identify a neighbourhood of $F(m_0)$ with a disk in $\mathbb{C}$ centered at the origin, with $F(m_0)$ identified with 0. We also extend P_0 in U by setting

$$P : U \longrightarrow \mathbb{C}$$

$$P(x_1, ..., x_4) = P_0(x_1 + ix_2, x_3 + ix_4).$$

It is clear that for $m \in U$ and $i = 1, ..., 4$

$$\frac{\partial P}{\partial x_i}(m) = \frac{\partial P_0}{\partial x_i}(x(m))$$

hence P is J_0-holomorphic.

Main Lemma 1. *Let M^4 be a Riemannian 4-manifold, N^2 a Riemannian 2-surface and $F : M^4 \longrightarrow N^2$ a harmonic morphism. We consider a critical point m_0 of F which we do not assume isolated. We denote by P_0 the symbol of F at m_0, assumed to be holomorphic for a parallel Hermitian complex structure J_0 on $T_{m_0} M^4$ and we extend J_0 to a neighbourhood*

of m_0 as explained above.
In a neighbourhood U of m_0, there exists an almost Hermitian structure J continuously defined on the regular points of F in U such that
1) J has the same orientation as J_0
2) F is pseudo-holomorphic w.r.t. J.
Moreover, for a point m in U

$$|J(m) - J_0(m)| \leqslant C_3|m|$$

for some positive constant C_3 independent of m.

Proof. We let $\Psi = F - P$. By definition of the symbol of F, there exist C_4, C_5 such that

$$\forall m \in U, \forall X \in T_m M |\Psi(m)| \leqslant C_4|m|^{k+1}, \quad |d\Psi(m)(X)| \leqslant C_5|m|^k\|X\| \tag{2.4}$$

We let (ϵ_1, ϵ_2) be a local positive orthonormal basis of N^2 in a neighbourhood of $u_0 := F(m_0)$. Denoting by j the complex structure on N^2, we have

$$\epsilon_2 = j\epsilon_1 \tag{2.5}$$

If m is a regular point of F, we define two unit orthogonal vectors e_1, e_2 in H_m such that

$$dF(e_1) = \lambda(m)\epsilon_1 \quad dF(e_2) = \lambda(m)\epsilon_2 \tag{2.6}$$

where $\lambda(m)$ denotes the dilation of F at m (see Def.8 and [B-W] pp. 46-47).
Next we pick an orthonormal basis (e_3, e_4) of V_m in a way that (e_1, e_2, e_3, e_4) is of the orientation defined by J_0. We define the almost complex structure J by setting

$$Je_1 = e_2 \quad Je_3 = e_4 \tag{2.7}$$

We first show that J_0e_1 is close to e_2; we set

$$J_0e_1 = ae_1 + be_2 + v \tag{2.8}$$

where $a, b \in \mathbb{R}$ and $v \in V_m$.
Since $a =< J_0e_1, e_1 >$, we get from (2.3)

$$|a| \leqslant C_2|m|^2 \tag{2.9}$$

Next we compute $dF(J_0e_1)$:

$$\begin{aligned} dF(J_0e_1) &= dP(J_0e_1) + d\Psi(J_0e_1) = jdP(e_1) + d\Psi(J_0e_1) \\ &= jdF(e_1) - jd\Psi(e_1) + d\Psi(J_0e_1) \end{aligned} \tag{2.10}$$

On the other hand, it follows from (2.8) that

$$\begin{aligned} dF(J_0e_1) &= adF(e_1) + bdF(e_2) \\ &= adF(e_1) + jbdF(e_1) \end{aligned} \tag{2.11}$$

by definition of e_1 and e_2 (see (2.6)).
Putting (2.10) and (2.11) together, we get

$$j(1-b)dF(e_1) = adF(e_1) + jd\Psi(e_1) - d\Psi(J_0e_1)$$

and using (2.6), we derive

$$|1-b|\lambda(m) = |adF(e_1) + jd\Psi(e_1) - d\Psi(J_0e_1)| \tag{2.12}$$

We already know that the right-hand side of (2.12) is a $\mathcal{O}(|m|^k)$; in order to show that $|1-b|$ is a $\mathcal{O}(|m|)$, we need to bound $\lambda(m)$ below.

Lemma 2. *There exists a $C_6 > 0$ such that, for m small enough,*

$$\lambda(m) \geqslant C_6|m|^{k-1}$$

Proof. First we notice that

$$\lambda(m) = \sup_{X \in T_mM^4, \|X\|=1} \|dF(m)X\| \tag{2.13}$$

Indeed, take a vector $X \in T_mM$ with $\|X\| = 1$. We split it into $X = X_v + X_h$ with X_v vertical and X_h horizontal. Then $\|X_v\|^2 + \|X_h\|^2 = 1$ and

$$\|dF(m)X\| = \|dF(m)X_h\| = \lambda(m)\|X_h\| \leqslant \lambda(m).$$

Since P_0 is of degree k, there exists C_7 such that for m small enough

$$\sup_{X \in T_mM^4, \|X\|=1} \|dP(m)X\| \geqslant C_7|m|^{k-1}.$$

It follows that, for $m \in U$ and $X \in T_mM^4$ with $\|X\| = 1$, we have

$$||dF(m)X|| = ||dP(m)X + d\Psi(m)X|| \geqslant ||dP(m)X|| - ||d\Psi(m)X||$$

$$\geqslant C_7|m|^{k-1} - C_5|m|^k$$

We take m small enough so that $C_5|m| \leqslant \frac{C_7}{2}$ and the lemma follows by taking $C_6 = \frac{C_7}{2}$. □

It follows from (2.12) and from Lemma 2 that

$$|b-1| \leqslant C_9|m| \tag{2.14}$$

for m small enough and some constant C_9.
To estimate $\|v\|$, we use (2.3) to write for m small enough

$$|\|J_0e_1\|^2 - 1| = |a^2 + b^2 + \|v\|^2 - 1| \leqslant C_2|m|^2 \tag{2.15}$$

Hence

$$\|v\|^2 \leqslant C_2|m|^2 + a^2 + |b^2 - 1|$$

and it follows from (2.9) and (2.14) that

$$\|v\| \leqslant C_{11}|m| \tag{2.16}$$

for some positive constant C_{11}.
We can now conclude. Since

$$\|Je_1 - J_0e_1\| = \|e_2 - J_0e_1\| \leqslant |a| + |b-1| + \|v\|$$

$\|Je_1 - J_0e_1\|$ is a $\mathcal{O}(|m|)$; similarly for $\|Je_2 - J_0e_2\|$.

We now prove that $\|Je_3 - J_0e_3\|$ is a $\mathcal{O}(|m|)$: there are no new ideas so we skip the details. We write

$$J_0e_3 = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4$$

Since (e_i) is an orthonormal basis,

$$|\alpha| = | < J_0e_3, e_1 > | \leqslant | < J_0e_1, e_3 > | + C_2|m|^2$$

using (2.3); it follows from the estimates above for J_0e_1 that α (and for the same reason β) is a $\mathcal{O}(|m|)$.
We also derive from (2.3) that

$$|\gamma| = | < J_0e_3, e_3 > | \leqslant C_2|m|^2$$

Now that we know that α, β and γ are $\mathcal{O}(|m|^2)$'s, we focus on δ and derive from (2.3)

$$|\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 1| = |\|J_0e_3\|^2 - 1| \leqslant C_2|m|^2$$

It follows that $|\delta^2 - 1|$ is an $\mathcal{O}(|m|^2)$, hence δ is either close to 1 or to -1: let us prove that δ is positive, using orientation arguments.
In a neighbourhood of m, we identify $\Lambda^4(M)$ with $\mathbb{R}$ so we can talk of signs of 4-vectors. If we denote by $\star$ the Hodge star operator, the sign of $e_1 \wedge J_0e_1 \wedge \star(e_1 \wedge J_0e_1)$ gives us the orientation of J_0 hence, by our assumption, it is of the same sign as $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We have seen that, close to m, J_0e_1 is close to e_2, hence $e_1 \wedge J_0e_1 \wedge \star(e_1 \wedge J_0e_1)$ and $e_1 \wedge J_0e_1 \wedge e_3 \wedge J_0e_3$ have the same sign; this latter 4-vector has the same sign as $\delta e_1 \wedge e_2 \wedge e_3 \wedge e_4$. It follows that δ is positive.

Hence $\|J_0e_3 - Je_3\|$ is a $\mathcal{O}(|m|)$ and so is $\|J_0e_4 - Je_4\|$, by identical arguments; this concludes the proof of the Main Lemma. □

2.3 Proof of Th.6 1): an isolated critical point

If m_0 is an isolated critical point, the almost complex structure J given by the Main Lemma is defined in $U \backslash \{m_0\}$ where U is a neighbourhood of m_0. At the point m_0, we put $J(m_0) = J_0$ and the Main Lemma tells us that the resulting almost complex structure is continuous.

2.4 Proof of Th.6 2):M is compact without boundary

2.4.1 Background: twistor spaces

We give here a brief sketch of Eells-Salamon's work with twistors ([Ee-Sal]); the reader can find a more detailed exposition in Chap. 7 of [B-W].
The *twistor space* $Z^+(M^4)$ (resp. $Z^-(M^4)$) of an oriented Riemannian 4-manifold (M^4, g) is the 2-sphere bundle defined as follows: a point in $Z^+(M^4)$ (resp. $Z^-(M^4)$) is of the form (J_0, m_0) where m_0 is a point in M^4 and J_0 is an orthogonal complex structure on $T_{m_0}M^4$ which preserves (resp. reverses) the orientation on $T_{m_0}M^4$. The twistor spaces $Z^\pm(M^4)$ admit the following almost complex structures $\mathcal{J}_\pm$.
We split the tangent space $T_{(J_0,m_0)}Z^\pm(M^4)$ into a horizontal space $\mathcal{H}_{(J_0,m_0)}$ and a vertical space $\mathcal{V}_{(J_0,m_0)}$. Since $\mathcal{H}_{(J_0,m_0)}$ is naturally identified with $T_{m_0}M^4$, we define $\mathcal{J}_\pm$ on $\mathcal{H}_{(J_0,m_0)}$ as the pull-back of J_0 from $T_{m_0}M^4$; the fibre above m_0 is an oriented 2-sphere so we define $\mathcal{J}_\pm$ on $\mathcal{V}_{(J_0,m_0)}$ as the opposite of the canonical complex structure on this 2-sphere. If S is an oriented 2-surface in M^4, it has a natural lift inside the twistor spaces: a point p in S lifts to the point (J_p, p) in $Z^+(M^4)$ (resp. $Z^-(M^4)$), where J_p is the orthogonal complex structure on T_pM^4 which preserves (resp. reverses) the orientation and for which the oriented plane T_pS is an oriented complex line.
Jim Eells and Simon Salamon proved

Theorem 7. *([Ee-Sal]) Let (M^4, g) be a Riemannian 4-manifold. A minimal surface in M^4 lifts into a $\mathcal{J}_+$-holomorphic (resp. $\mathcal{J}_-$-holomorphic) curve in $Z^+(M^4)$ (resp. $Z^-(M^4)$). Conversely, every non vertical $\mathcal{J}_\pm$-holomorphic curve in $Z^\pm(M^4)$ is the lift of a minimal surface in M^4.*

2.4.2 Convergence of the twistor lifts of regular fibres

We assume M^4 to be oriented: Th.6 is local so if M^4 is not oriented, we endow a ball centered at m_0 with an orientation given by the complex structure J_0 on $T_{m_0}M^4$ defined by the symbol (see §2.2.2). From now on we drop the superscript $+$ and we write $Z(M^4)$ for $Z^+(M^4)$.
We denote $u_0 = F(m_0)$ and we let (u_n) be a sequence of regular values of F which converges to u_0. The preimages of the u_n's are smooth compact closed 2-submanifolds of M^4. For every positive integer n, we let

$$S_n = F^{-1}(u_n).$$

Lemma 3. *The S_n's all have the same area.*

Proof. The singular values of F are discrete so we can assume that the S_n's are all deformation of one another; moreover they are all minimal. It follows from the formula for the first variation of area that if (Σ_t), $t \in [0,1]$, is a family of minimal surfaces without boundary in a compact manifold, $\frac{d}{dt}area(\Sigma_t) = 0$, hence $area(\Sigma_t)$ is constant, for $t \in [0,1]$. □

We denote by $\tilde{S}_n$ the lift of S_n into $Z^+(M^4)$: Th. 7 tells us that they are $\mathcal{J}$-holomorphic curves. Moreover we have

Lemma 4. *There exists a constant C such that, for every positive integer n,*

$$area(\tilde{S}_n) \leqslant C.$$

Proof. We parametrize the $\tilde{S}_n$'s by maps

$$\gamma_n : S_n \longrightarrow \tilde{S}_n$$

We let (e_1, e_2) be an orthonormal basis of the tangent bundle TS_n and we denote by $\tilde{e}_1, \tilde{e}_2$ their lift in $Z(M^4)$. For $i = 1, 2$, we split $\tilde{e}_i$ into vertical and horizontal components,

$$\tilde{e}_i = \tilde{e}_i^h + \tilde{e}_i^v$$

We write the area element of $\tilde{S}_n$:

$$\|\tilde{e}_1 \wedge \tilde{e}_2\| \leqslant \|\tilde{e}_1^h \wedge \tilde{e}_2^h\| + \|\tilde{e}_1^h \wedge \tilde{e}_2^v\| + \|\tilde{e}_1^v \wedge \tilde{e}_2^h\| + \|\tilde{e}_1^v \wedge \tilde{e}_2^v\| \tag{2.17}$$

Integrating (2.17) and using Cauchy-Schwarz inequality, we get

$$area(\tilde{S}_n) \leqslant area(S_n) + 2\sqrt{area(S_n)}\sqrt{\int_{S_n} \|\nabla\gamma_n\|^2} + \int_{S_n} \|\nabla\gamma_n\|^2$$

where ∇ denote the connection on $Z(M^4)$ induced by the Levi-Civita connection on M^4.

Lemma 5. *There exists a constant A such that, for every positive n*

$$\int_{S_n} \|\nabla\gamma_n\|^2 \leqslant A$$

Proof. We need to introduce a few notations to give a formula for the integral in Lemma 5. For every n, we let NS_n be the normal bundle of S_n in M^4 and we endow it with a local orthonormal basis (e_3, e_4). We denote by R the curvature tensor of (M^4, g) and we put

$$\Omega^T =< R(e_1, e_2)e_1, e_2 > \qquad \Omega^N =< R(e_1, e_2)e_3, e_4 >$$

Finally we let $c_1(NS_n)$ be the degree of NS_n i.e. the integral of its 1st Chern class; it changes sign with the orientation of M^4. Note that other authors (e.g. [C-T]) denote it by $\chi(NS_n)$, by analogy with the Euler characteristic.
We denote by dA the area element of S_n and we derive from [C-T] (see also [Vi 2])

$$\frac{1}{2}\int_{S_n} \|\nabla\gamma_n\|^2 = -\chi(S_n) - c_1(NS_n) + \int_{S_n} \Omega^T dA + \int_{S_n} \Omega^N dA \tag{2.18}$$

The critical values of F are isolated, hence the regular fibres all have the same homotopy type and the same homology class $[S_n]$ in $H_2(M^4, \mathbb{Z})$. In particular, $|\chi(S_n)|$ does not depend on n. The S_n's are embedded hence $c_1(NS_n)$ is equal to the self-intersection number $[S_n].[S_n]$ which does not depend on n either.
Since M^4 is compact, the expression $| < R(u,v)w, t > |$ has an upper bound for all the 4-uples of unit vectors (u, v, w, t). It follows that the integrals in Ω^T and Ω^N in (2.18) have a common bound in absolute value.
In conclusion, all the terms in the RHS of (2.18) are bounded in absolute value uniformely in n. ∎

Lemma 4 follows immediately. □

Thus the $\tilde{S}_n$'s are $\mathcal{J}$-holomorphic curves of bounded area in $Z(M^4)$: Gromov's result ([Gro]) ensures that they admit a subsequence which converges in the sense of cusp-curves to a $\mathcal{J}$-holomorphic curve C.

Lemma 6. *We denote by $\pi : Z(M^4) \longrightarrow M^4$ the natural projection. Then*

$$\pi(C) = F^{-1}(u_0).$$

Proof. The map $\pi \circ F$ is continuous, so it is clear that $\pi(C) \subset F^{-1}(u_0)$.
To prove the reverse inclusion, we take a point $p \in F^{-1}(u_0)$ and we claim

Claim 1. *There exists a subsequence $(u_{s(n)})$ of (u_n) and a sequence of points (p_n) of M^4 converging to p with*

$$F(p_n) = u_{s(n)}$$

for every positive integer n.

Indeed, if Claim 1 was not true, we would have the following

Claim 2. *$\exists \epsilon > 0$ such that $\forall n \in \mathbb{N}^*$ and $\forall m \in M^4$ with $F(m) = u_n$, we have*

$$d(m,p) > \epsilon.$$

If Claim 2 was true, the set $F(B(p,\epsilon))$ would contain u_0 but would not be a neighbourhood of u_0, a contradiction of the fact that a harmonic morphism is open ([Fu],[B-W] p.112).
So Claim 1 is true: if we denote by (J_n, p_n) the pullback of the p_n's in the twistor lifts $\tilde{S}_n$, they admit a subsequence which converges to a point $(\hat{J}, p)$, for some $\hat{J}$ in the twistor fibre above p. Clearly $(\hat{J}, p)$ belongs to C, hence p belongs to $\pi(C)$ and Lemma 6 is proved. □

Their are a finite number of points $p_1, ..., p_k \in M^4$ and positive integers $q_1, ..., q_k$ such that the curve C can be written

$$C = \Gamma + \sum_{i=1}^{k} q_i Z_{p_i} \tag{2.19}$$

where Γ is a $\mathcal{J}$-holomorphic curve with no vertical components and the Z_{p_i}'s are the twistor fibres above the p_i's. It follows from Lemma 6 that

$$\pi(\Gamma) = F^{-1}(u_0).$$

We derive that $F^{-1}(u_0)$ is a minimal surface possibly with branched points and having Γ as its twistor lift.
Note that the presence of twistor fibres in (2.19) is to be expected: when a sequence of smooth minimal surfaces converges to a minimal surface with singularities, its twistor lifts can experience bubbling off of twistor fibres above singular points (see [Vi2] for a more detailed discussion of this phenomenon). However, in the present case, the Main Lemma excludes such bubbling-off in a neighbourhood of m_0:

Lemma 7. *There exists an $\epsilon > 0$ such that, if p_i is one of the points appearing in (2.19),*

$$dist(m_0, p_i) > \epsilon$$

Proof. Since the p_i's are finite in number, it is enough to prove that m_0 is not one of them. The almost complex structure J_0 appearing in the Main Lemma does not necessarily preserve the metric outside of m_0; so we introduce the bundle $\mathcal{C}$ of *all* the complex structures on TM^4 which preserve the orientation. It contains the bundle $Z(M^4)$ and embeds into the bundle $GL(TM^4)$. We denote by $d_{\mathcal{C}}$ the distance on $\mathcal{C}$ induced by the metric on $GL(TM^4)$ and by d_{M^4} the distance in M^4 and we prove :

Lemma 8. $\forall \epsilon > 0 \quad \exists \eta > 0$ *such that*

$$d_{M^4}(m, m_0) < \eta \Rightarrow d_{\mathcal{C}}[(J(m), m), (J_0, m_0)] < \epsilon$$

Proof. $d_{\mathcal{C}}[(J(m), m), (J_0, m_0)]$

$$\leqslant d_{\mathcal{C}}[(J(m), m), (J_0(m), m)] + d_{\mathcal{C}}[(J_0(m), m), (J_0, m_0)] \tag{2.20}$$

We bound the first term in (2.20) using the Main Lemma; the second term is bounded because $J_0 : U \longrightarrow \mathcal{C}$ is continuous. □

If m is a regular point of F, we denote by $\gamma(m)$ the point above m in the twistor lift of $F^{-1}(F(m))$; in the Main Lemma, we defined the almost complex structure $J(m)$. The tangent plane to the fibre at m is a complex line for both $\gamma(m)$ and $J(m)$; since $\gamma(m)$ and $J(m)$ both preserve the orientation, it follows that $\gamma(m) = \pm J(m)$. We can get rid of the $\pm$ by a connectivity argument and derive an $s \in \{-1, +1\}$ such that for every regular point m of F near m_0,

$$\gamma(m) = sJ(m) \tag{2.21}$$

We rewrite Lemma 8: $\forall \epsilon > 0 \quad \exists \eta > 0$ such that for a regular point m,

$$d_M(m, m_0) < \eta \Rightarrow d_{\mathcal{C}}[(\gamma(m), m), (sJ_0, m_0)] < \epsilon \tag{2.22}$$

If the whole twistor fibre $Z_{m_0}M^4$ was included in C, it would be in the closure of the union of the twistor lifts of the regular fibres of F in a neighbourhood of m_0: we see from (2.22) that this is impossible. This concludes the proof of Lemma 7. □

2.4.3 Construction of the almost complex structure

We now construct a local section of $Z(M^4)$, for which F is holomorphic. As in [Vi1], we work first on the space $\mathbb{P}(Z(M^4))$ obtained by taking the quotient of each twistor fibre by its antipody; if J is an element of $Z(M^4)$, we denote by $\bar{J}$ its image in $\mathbb{P}(Z(M^4))$.
If m is a regular point of F, there are 2 complex structures, J_1 and J_2, on $T_{m_0}M^4$ for which the unoriented planes V_m and H_m are complex lines. These two complex structures verify $J_1 = -J_2$, hence they define the same point, denoted $\bar{J}(m)$, in $\mathbb{P}(Z_m(M^4))$. To extend this section of $\mathbb{P}(Z(M^4))$ above $F^{-1}(u_0)$, we state

Lemma 9. *There exists an $\epsilon > 0$ such that every $m \in B(m_0, \epsilon) \cap F^{-1}(u_0)$ has either a single preimage in Γ or exactly two antipodal preimages in Γ.*

Proof. We let ϵ be a number satisfying Lemma 7 and we pick $m \in B(m_0, \epsilon) \cap F^{-1}(u_0)$. Since, by defnition, Γ has no vertical component it meets $Z_m(M^4)$ at a discrete number of points. Let us assume that J_1 and J_2 are two different elements of $\Gamma \cap Z_m(M^4)$. There exist two non vertical possibly branched disks Δ_1 and Δ_2 in Γ containing (J_1, m) and (J_2, m) respectively. Each one of the two Δ_i's is the twistor lift of a (possibly branched) disk D_i of $F^{-1}(u_0)$. The disks D_1 and D_2 meet at m: if they have different tangent planes at m, this implies that m is a singular point of $F^{-1}(u_0)$. Since $F^{-1}(u_0)$ is a closed minimal surface, its singular points are discrete so we can make ϵ small enough so that there is no singular point in $F^{-1}(u_0) \cap B(m_0, \epsilon)$ except for possibly m_0.
So we assume that m_0 is a singular point of $F^{-1}(u_0)$. Because the symbol is J_0 holomorphic, all planes tangent to m_0 at $F^{-1}(u_0)$ are J_0-complex lines and it follows that $J_1 = -J_2 = \pm J_0$. □

We denote by $\bar{\Gamma}$ the projection of Γ in $\mathbb{P}(Z(M^4))$ and by $\bar{J}$ the local section of $\bar{\Gamma}$ given by Lemma 9.

Lemma 10. *There exists a small $\epsilon > 0$ such that the map*

$$B(m_0, \epsilon) \longrightarrow \mathbb{P}(Z(M^4))$$

$$m \mapsto \bar{J}(m)$$

is continuous.

Proof. Since $\bar{J}$ is continuous above $U \backslash F^{-1}(u_0)$, we consider a sequence of points (p_n) in M^4 converging to a p_0 with $F(p_0) = u_0$. It is enough to the consider two cases
1st case: all the $F(p_n)$'s are regular values
2nd case: for every n, $F(p_n) = u_0$.
If (p_n) is a general sequence, we extract subsequences of the form 1) or 2).
1st case - For every n, $v_n = F(p_n)$ is a regular value of F.
i) First assume that $u_n = v_n$ for every n. Since Γ is the limit of the twistor lifts of the $F^{-1}(u_n)$ the sequence $(\bar{J}(p_n), p_n)$ converges to a point $(\bar{K}, p_0)$ in $\bar{\Gamma}$; Lemma 9 ensures that $\bar{K} = \bar{J}(p_0)$.
ii) In the general case, the v_n's converge to u_0 so we can proceed with the v_n's as we did with the u_n's and derive that the twistor lifts of the $F^{-1}(v_n)$'s converge in the sense of Gromov to the twistor lift of $F^{-1}(u_0)$ and conclude as in i).
2nd case For every n, $F(p_n) = u_0$. We denote by $\bar{\pi}$ the natural projection from $\mathbb{P}(Z(M^4))$ to M^4. Lemma 9 ensures that $\bar{\pi}$ restricts to a continuous bijection from $\bar{\Gamma} \cap \bar{\pi}^{-1}(\bar{B}(m_0, \frac{\epsilon}{2}))$ to $F^{-1}(u_0) \cap \bar{B}(m_0, \frac{\epsilon}{2})$; since these spaces are compact and Hausdorff, a continuous bijection between them is a homeomorphism (see for example [Han] p. 45). It follows that, if the p_n's converge to p_0, their preimages in $\bar{\Gamma}$ converge to the preimage of p_0; in other words, the $\bar{J}(p_n)$'s converge to $\bar{J}(p_0)$. □

We conclude as in [Vi1]. We lift $\bar{J}$ above the set of regular points by taking for J the one complex structure on $T_m M$ which renders dF holomorphic at that point - this

requirement defines it uniquely on the horizontal space H_m and since, the orientation of J is given, there is also a unique possibility for J on V_m. By the same argument as in [Vi1], this extends to the entire $B(m_0, \epsilon)$.
This concludes the proof of Th.6

2.5 Proof of Prop. 8

We begin by reproducing part of Wood's arguments ([Wo]).
We let m be a regular point of F and we denote by V_m (resp. H_m) the vertical (resp. horizontal) space at m. We let S_0V_m be the set of symmetric trace-free homomorphisms of V_m and we define the Weingarten map

$$A : H_m \longrightarrow S_0V_m$$

$$X \mapsto (U \mapsto \nabla^V_U X)$$

where $\nabla^V_U X$ denotes the vertical projection of $\nabla_U X$.
At a regular point m, we denote by J_+ (resp. J_-) the Hermitian structure on T_mM^4 w.r.t. which $dF : T_mM^4 \longrightarrow T_{F(m)}N^2$ is $\mathbb{C}$-linear and which preserves (resp. reverses) the orientation on T_mM^4. If M^4 is Einstein, Wood proves in Prop. 3.2 that all horizontal vectors X verify

$${}^tA \circ A(J_\pm X) = J_\pm({}^tA \circ A)(X).$$

If M^4 is not Einstein, we follow his proof to derive the existence of C_{11} such that, for every unit horizontal vector X tangent to a regular point of F in K,

$$\|{}^tA \circ A(J_\pm X) - J_\pm({}^tA \circ A)(X)\| \leqslant C_{11} \tag{2.23}$$

We now put $T = e_1$ and we complete it into an orthonormal basis (e_1, e_2) of V_m; we pick an orthonormal basis (e_3, e_4) of H_m such that the almost complex structures verify

$$e_2 = J_+e_1 = -J_-e_1 \qquad e_4 = J_+e_3 = J_-e_3 \tag{2.24}$$

We let E_1 and E_2 be the following elements of S_0V_m defined by their matrices in the base (e_1, e_2).

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We write the matrix of A in the bases (e_3, e_4) and (E_1, E_2)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$a = - < \nabla_{e_1} e_1, e_3 > \qquad b = - < \nabla_{e_1} e_1, e_4 > \tag{2.25}$$

$$c = - < \nabla_{e_1} e_2, e_3 > \qquad d = - < \nabla_{e_1} e_2, e_4 > \tag{2.26}$$

The homomorphisms J_+ and J_- coincide on the basis (e_3, e_4) (see (2.24)); we compute

$$({}^tA \circ A)J_\pm - J_\pm({}^tA \circ A) = \begin{pmatrix} 2(ab+cd) & b^2+d^2-(a^2+c^2) \\ b^2+d^2-(a^2+c^2) & -2(ab+cd) \end{pmatrix}$$

and we derive from (2.23)

$$|ab+cd| \leqslant C_{11} \qquad |b^2+d^2-(a^2+c^2)| \leqslant C_{11} \tag{2.27}$$

We take J to be J_+ or J_- and we write the Euclidean norm

$$\|\nabla_{e_1} J\|^2 = \sum_{i,j=1,\dots,4} < (\nabla_{e_1} J)e_i, e_j >^2 \tag{2.28}$$

$$= \sum_{i,j=1,\dots,4} (< \nabla_{e_1}(Je_i), e_j > - < J\nabla_{e_1} e_i, e_j >)^2 \tag{2.29}$$

$$= \sum_{i,j=1,\dots,4} (< \nabla_{e_1}(Je_i), e_j > + < \nabla_{e_1} e_i, Je_j >)^2 \tag{2.30}$$

It is enough to take e_i vertical and e_j horizontal in (2.30):

Lemma 11. $\|\nabla_{e_1} J\|^2 = 2 \sum_{\substack{1 \leqslant i \leqslant 2 \\ 3 \leqslant j \leqslant 4}} (< \nabla_{e_1}(Je_i), e_j > + < \nabla_{e_1} e_i, Je_j >)^2.$

Proof. If e_i and e_j are both horizontal or both vertical, Prop. 2.5.16 i) of [B-W] yields

$$< \nabla_{e_1}(Je_i), e_j > = < J\nabla_{e_1} e_i, e_j > \tag{2.31}$$

Note that Baird-Wood's Prop. 2.5.16 is about horizontal vectors, but its proof works identically for vertical vectors.
Assume now that e_i is horizontal and e_j is vertical:

$$< \nabla_{e_1}(Je_i), e_j > + < \nabla_{e_1} e_i, Je_j > = - < Je_i, \nabla_{e_1} e_j > - < e_i, \nabla_{e_1}(Je_j) > \tag{2.32}$$

Putting together (2.30), (2.31) and (2.32) completes the proof of Lemma 11. □

We use the values given for the $J_\pm$ in (2.24) to derive

$$\frac{1}{2}\|\nabla_{e_1} J_\pm\|^2 = (\pm < \nabla_{e_1} e_2, e_3 > + < \nabla_{e_1} e_1, e_4 >)^2$$
$$+(\pm < \nabla_{e_1} e_2, e_4 > - < \nabla_{e_1} e_1, e_3 >)^2$$
$$+(\pm < \nabla_{e_1} e_1, e_3 > - < \nabla_{e_1} e_2, e_4 >)^2$$
$$+(\pm < \nabla_{e_1} e_1, e_4 > + < \nabla_{e_1} e_2, e_3 >)^2 \tag{2.33}$$

We rewrite (2.33) in terms of the coefficients a, b, c, d of the matrix A introduced above (see (2.25) and (2.26)); we get after a short computation

$$\|\nabla_{e_1} J_+\|^2 = 4[(a-d)^2 + (b+c)^2] = 4[a^2 + b^2 + c^2 + d^2 - 2(ad - bc)] \tag{2.34}$$

$$\|\nabla_{e_1} J_-\|^2 = 4[(a+d)^2 + (b-c)^2] = 4[a^2 + b^2 + c^2 + d^2 + 2(ad - bc)] \tag{2.35}$$

hence

$$\|\nabla_{e_1} J_+\|^2 \|\nabla_{e_1} J_-\|^2 = 16[(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2] \tag{2.36}$$

We now bound (2.36) using (2.27). To this effect we put

$$a = R_1 \cos\theta \qquad c = R_1 \sin\theta \qquad b = R_2 \cos\alpha \qquad d = R_2 \sin\alpha \tag{2.37}$$

and we rewrite (2.36) as

$$\frac{1}{16}\|\nabla_{e_1} J_+\|^2 \|\nabla_{e_1} J_-\|^2 = (R_1^2 + R_2^2)^2 - 4R_1^2 R_2^2 \sin^2(\theta - \alpha) \tag{2.38}$$

$$= (R_1^2 + R_2^2)^2 - 4R_1^2 R_2^2 + 4R_1^2 R_2^2 \cos^2(\theta - \alpha) \tag{2.39}$$

$$= (R_1^2 - R_2^2)^2 + 4R_1^2 R_2^2 \cos^2(\theta - \alpha) \tag{2.40}$$

We now rewrite (2.27) as

$$|R_1 R_2 \cos(\theta - \alpha)| \leqslant C_{11} \qquad |R_1^2 - R_2^2| \leqslant C_{11} \tag{2.41}$$

and this allows us to bound (2.40) and conclude the proof of of Prop. 8.

Chapter 3

Harmonic morphisms on $\mathbb{S}^4$

In this paper we study examples of harmonic morphisms due to Burel from $(\mathbb{S}^4, g_{k,l})$ into $\mathbb{S}^2$ where $(g_{k,l})$ is a family of conformal metrics on $\mathbb{S}^4$. To do this construction we define two maps, F from $(\mathbb{S}^4, g_{k,l})$ to $(\mathbb{S}^3, \bar{g_{k,l}})$ and $\phi_{k,l}$ from $(\mathbb{S}^3, \bar{g_{k,l}})$ to $(\mathbb{S}^2, can)$; the two maps are both horizontally conformal and harmonic. Let $\Phi_{k,l} = \phi_{k,l} \circ F$. It follows from Baird-Eells that the regular fibres of $\Phi_{k,l}$ for every k, l are minimal. If $|k| = |l| = 1$, the set of critical points is given by the preimage of the north pole : it consists in two 2-spheres meeting transversally at 2 points. If $k, l \neq 1$ the set of critical points are the preimages of the north pole (the same two spheres as for $k = l = 1$ but with multiplicity l) together with the preimage of the south pole (a torus with multiplicity k).

3.1 Introduction

A harmonic morphism $F : M \longrightarrow N$ between two Riemannian manifolds (M, g) and (N, g) is a map which pulls back local harmonic functions on N to local harmonic functions on M. Although harmonic morphisms can be traced back to Jacobi, their study in modern times was initiated by Fuglede and Ishihara who characterized them using the notion of horizontal weak conformality, or semiconformality:

Definition 9. *(see [B-W] p.46)*

Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds and let $x \in M$. Then F is called horizontally weakly conformal at x if either

1) $dF_x = 0$

2) dF_x *maps the space* $Ker(dF_x)^\perp$ *conformally onto* $T_{F(x)}N$, *i.e. there exists a number* $\lambda(x)$ *called the dilation of* F *at* x *such that*

$$\forall X, Y \in Ker(dF_x)^\perp, h(dF_x(X), dF_x(X)) = \lambda^2(x) g(X, Y).$$

The space $Ker(dF_x)$ *(resp.* $Ker(dF_x)^\perp$*) is called the vertical (resp. horizontal) space at* x.

Fuglede and Ishihara proved independently

Theorem 8. *([Fu],[Is])*
Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. The following two statements are equivalent:

1) For every harmonic function $f : V \longrightarrow \mathbb{R}$ defined on an open set V of N, the function $f \circ F$ defined on the open set $F^{-1}(V)$ of M is harmonic.

2) The map F is harmonic and horizontally weakly conformal.
Such a map is called a harmonic morphism

When the target is 2-dimensional, Baird and Eells proved.

Theorem 9. *([B-E])*
Let $F : (M^m, g) \longrightarrow (N^2, h)$ be a smooth non constant horizontally weakly conformal map between a Riemannian manifold (M^m, g) and a Riemannian 2-surface (N^2, h). Then F is harmonic (hence a harmonic morphism) if and only if the fibres of F at regular points are minimal submanifolds of M.

In Makki-Ville ([Ma-Vi]) we extend Th.9 to the singular fibres if M is compact. There is no non constant harmonic morphisms from $(\mathbb{S}^4, can)$ to $\mathbb{S}^2$ ([Wo,Vi]). So Burel [Bu] endows $\mathbb{S}^4$ with metrics g conformal to the canonical metric σ, for which he constructed many harmonic morphisms from $(\mathbb{S}^4, g)$ to $\mathbb{S}^2$.

3.2 Motivation

Let C be a complex curve in a complex compact manifold M of complex dimension two. The *adjunction formula* [G-H] which relates the tangent bundle, normal bundle and homology class of a complex curve in $\mathbb{C}P^2$ is given by

$$c_1(TC) + c_1(NC) = c_1\left(T\mathbb{C}P^2\right)|_C$$

and $c_1(TC) + c_1(NC)$ depends only on the homology class of C in $\mathbb{C}P^2$.
In particular, let (C_n) be a family of complex curves in $\mathbb{C}P^2$ such that, for $n \neq 0$, C_n is smooth and $C_n \longrightarrow C_0$ and C_0 has one branch point. Then

$$c_1(TC_n) + c_1(NC_n) = c_1(TC_0) + c_1(NC_0) \tag{3.1}$$

Exemple 1. *Let (C_ϵ) given by $z_1 z_2 = \epsilon z_0^2$ be a family of complex curves in $\mathbb{C}P^2$ and (C_0) given by $z_1 z_2 = 0$ the union of $S_1 = \{z_1 = 0\}$ and $S_2 = \{z_2 = 0\}$. Then we have:*

$$c_1(TC_\epsilon) = 2$$

because C_ϵ is defined by a polynomial of degree two (C_ϵ is a sphere) , and

$$c_1(NC_\epsilon) = [C_\epsilon] \cdot [C_\epsilon] = 4.$$

because it is embedded and of degree two.
On the other hand, since C_0 is the union of two spheres,

$$c_1\left(TC_0\right) = 2+2 = 4,$$

and since C_0 has a positive self-intersection point :

$$c_1\left(NC_0\right) = \left[C_0\right]\cdot\left[C_0\right] - 2 = \left[C_\epsilon\right]\left[C_\epsilon\right] - 2 = 4-2 = 2.$$

So that

$$c_1\left(TC_\epsilon\right) + c_1\left(NC_\epsilon\right) = 6$$

and

$$c_1\left(TC_0\right) + c_1\left(NC_0\right) = 6.$$

By contrast, let M^4 be an oriented manifold.
We ask here what happens if (Σ_n) is a sequence of minimal surfaces which degenerates to (Σ_0) with a branch point ? Here (Σ_n) verify ([Vi2]),[C-T])

$$c_1\left(T\Sigma_n\right) + c_1\left(N\Sigma_n\right) \leqslant c_1\left(T\Sigma_0\right) + c_1\left(N\Sigma_0\right) \tag{3.2}$$

If we change the orientation on M^4, but not on the $\Sigma_n' s$, $c_1\left(T\Sigma_n\right)$ is unchanged and $c_1\left(N\Sigma_n\right)$ becomes $-c_1\left(N\Sigma_n\right)$. Hence (3.2) yields the following

$$c_1\left(T\Sigma_n\right) - c_1\left(N\Sigma_n\right) \leqslant c_1\left(T\Sigma_0\right) - c_1\left(N\Sigma_0\right). \tag{3.3}$$

When a singularity appears, we cannot have equality both in (3.2) and (3.3) because $c_1(T\Sigma_0) \neq c_1(T\Sigma_n)$.
In particular the complex curves C_n's converging in $\mathbb{C}P^2$ to C_0 as above satisfy the strict inequality (3.3):

$$c_1\left(TC_n\right) - c_1\left(NC_n\right) < c_1\left(TC_0\right) - c_1\left(NC_0\right). \tag{3.4}$$

Now if we change the orientation on $\mathbb{C}P^2$, the (C_n) will still be minimal surfaces in $\mathbb{C}P^2$ and they will verify for the new orientation

$$c_1\left(TC_n\right) + c_1\left(NC_n\right) < c_1\left(TC_0\right) + c_1\left(NC_0\right).$$

So we ask

Question 1. *When do we have a strict inequality* **both** *in (3.2) or (3.3) for the same orientation ?*

Exemple 2. *Consider the Burel map $\Phi_{1,1}$ and let (Σ_n) be a family of regular fibres in $\mathbb{S}^4$ which converges to the singular fibre Σ_0. We shall see below that the Σ_n's are embedded tori and that Σ_0 is the union of two spheres S_1 and S_2 with two tranverse intersection points of opposite signs. We have*

$$c_1\left(T\Sigma_n\right) = 0$$

and

$$c_1\left(N\Sigma_n\right) = [\Sigma_n] \cdot [\Sigma_n] = 0.$$

On the other hand:

$$c_1\left(T\Sigma_0\right) = 4,$$

and

$$c_1\left(N\Sigma_0\right) = [\Sigma_n] \cdot [\Sigma_n] - 2(1-1) = 0.$$

Thus

$$c_1\left(T\Sigma_n\right) \pm c_1\left(N\Sigma_n\right) = 0$$

and

$$c_1\left(T\Sigma_0\right) \pm c_1\left(N\Sigma_0\right) = 4.$$

3.3 Burel's construction

Burel was building upon previous constructions on product of spheres by Baird and Ou ([B-O]). He constructs a horizontally conformal map $\Phi_{k,l}$ with $k,l \in \mathbb{N}^*$ from $\mathbb{S}^4$ into $\mathbb{S}^2$ by the composition of two horizontally conformal maps F from $\mathbb{S}^4$ into $\mathbb{S}^3 = \mathbb{S}^0 * \mathbb{S}^2$ and $\phi_{k,l}$ from $\mathbb{S}^3 = \mathbb{S}^1 * \mathbb{S}^1$ into $\mathbb{S}^2$.
The key-point of this construction is the change of variable that allows to identify the joint $\mathbb{S}^0 * \mathbb{S}^2$ and the joint $\mathbb{S}^1 * \mathbb{S}^1$.
First we are going to define the Hopf fibration H from $\mathbb{S}^3$ into $\mathbb{S}^2$ and then use it to define the map F from $\mathbb{S}^4$ into $\mathbb{S}^3$.

Definition 10. *The Hopf fibration $H : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ of the 3-sphere over the 2-sphere is defined by*

$$H(z_0, z_1) = (|z_0|^2 - |z_1|^2, 2z_0z_1). \tag{3.5}$$

Let $x = (\cos t\; e^{ia},\; \sin t\; e^{ib})$ a point in $\mathbb{S}^3$ where $t \in [0, \pi/2]$ and $a, b \in [0, 2\pi]$ then

$$\begin{aligned} H(x) &= (\cos^2 t - \sin^2 t, 2\cos t \sin t e^{i(a+b)}) \\ &= (\cos 2t, \sin 2t e^{i(a+b)}) \end{aligned} \tag{3.6}$$

We define the map $F : \mathbb{S}^4 \to \mathbb{S}^3$ for $s \in [0, \pi],\; t \in [0, \pi/2]$ and $a, b \in [0, 2\pi]$ by

$$\begin{aligned} F\left(\cos s, \sin s\; \left(\cos t\; e^{ia},\; \sin t\; e^{ib}\right)\right) &= (\cos\alpha(s), \sin\alpha(s)\; H(x)) \\ &= \left(\cos\alpha(s), \sin\alpha(s)\cos 2t, \sin\alpha(s)\sin 2t\; e^{i(a+b)}\right). \end{aligned} \tag{3.7}$$

where α is a increasing regular function such that $\alpha(0) = 0$ and $\alpha(\pi) = \pi$,
with $\alpha(s)$ chosen so that F is semi-conformal *i.e.*

$$\alpha(s) = 2\arctan\left(\tan^2\left(\frac{s}{2}\right)\right).$$

Now for s fixed we have a geodesic sphere centred at the north pole of $\mathbb{S}^4$ of radius $\sin s$. The map F sends it to a geodesic sphere centered at the north pole of $\mathbb{S}^3$ of radius $\sin\alpha(s)$.

Between the 3-sphere and the 2-sphere the map F is the $Hopf$ map.
We now define the map $\varphi_{k,l}$ from $\mathbb{S}^3$ to $\mathbb{S}^2$. We need to define a new coordinate system on an open dense subset of $\mathbb{S}^3$ which allows us to go from $\mathbb{S}^3 = \mathbb{S}^0 * \mathbb{S}^2$ into $\mathbb{S}^3 = \mathbb{S}^1 * \mathbb{S}^1$, and this by supposing :

$$\cos\alpha(s) + i\sin\alpha(s)\cos 2t = \cos u(s,t)\ e^{i\psi(s,t)} \tag{3.8}$$

$$\sin\alpha(s)\sin 2t\ e^{i(a+b)} = \sin u(s,t)\ e^{i(a+b)}. \tag{3.9}$$

By changing the variable the point now is of the form

$$\left(\cos u(s,t)\ e^{i\psi(s,t)}, \sin u(s,t)\ e^{i(a+b)}\right) \text{ in } \mathbb{S}^3. \tag{3.10}$$

For simplification we write u, ψ, α instead of $u(s,t), \psi(s,t), \alpha(s,t)$.
Now let $\beta : [0, \frac{\pi}{2}] \longrightarrow [0, \pi]$ be a regular function of u such that

$$\beta(0) = 0 \text{ and } \beta\left(\frac{\pi}{2}\right) = \pi.$$

Note that the domain of β is $\left[0, \frac{\pi}{2}\right]$ and not $[0, \pi]$ as stated in [Bu].
In the new coordinate system, we define the application $\phi_{k,l} : \mathbb{S}^3 \to \mathbb{S}^2$ by:

$$\phi_{k,l}\left(\cos u\ e^{i\psi}, \sin u\ e^{i(a+b)}\right) = \left(\cos\beta(u), \sin\beta(u)\ e^{i(k\psi + l(a+b))}\right) \tag{3.11}$$

where $\beta(u)$ is chosen so that $\phi_{k,l}$ is horizontally conformal.
For this, β must satisfy the following equation:

$$\frac{\beta'_u}{\sin\beta} = \sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}}.$$

This equation has an explicit solution given by ([B-O]) see next section

$$\beta(u) = 2\arctan\left\{\left|\frac{l - p(u)}{l + p(u)}\right|^{\frac{l}{2}} \left|\frac{k + p(u)}{k - p(u)}\right|^{\frac{k}{2}}\right\} \tag{3.12}$$

with $p(u) = \sqrt{k^2\sin^2 u + l^2\cos^2 u}$.
Notice that the absolute value in the equation is missing in [Bu].

3.4 Computation of β

We now compute the function β and prove (3.12) following the hints of [B-O]. We begin by quoting a result of [B-O].

Lemma 12. *Let $F : (r_1 S^1) \times .... \times (r_p S^p) \longrightarrow aS^1$ be the map from the product of p circles of radius $r_1, ..., r_p$, respectively, into a circle of radius a, given by*

$$F(r_1 e^{i\theta_1}, ..., r_p e^{i\theta_p}) = a e^{i(k_1\theta_1 + ... + k_p\theta_p)},\ \textit{for integers } k_1, ..., k_p. \tag{3.13}$$

Then F is a harmonic morphism with dilation λ given by

$$\lambda^2 = a^2\left(\frac{k_1^2}{r_1^2} + ... + \frac{k_p^2}{r_p^2}\right) \tag{3.14}$$

We define a map $\phi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ as follows:

$$\mathbb{S}^3 \ni \left(\cos u e^{i\psi}, \sin u e^{iA}\right) \longrightarrow (\cos \beta(u), \sin \beta(u) e^{i(k\psi + lA)})$$

where $\psi, A \in [0, 2\pi]$, k, l are non-zero integers and $u \in [0, \pi/2]$.
We begin by solving the horizontal conformality condition for ϕ.
For fixed $u_0 \in (0, \pi/2)$, by lemma 12, the restriction of ϕ to the product of circles:

$$\cos u S^1 \times \sin u S^1 \longrightarrow \sin \beta S^1$$

$$(\cos u e^{i\psi}, \sin u e^{iA}) \longrightarrow \sin \beta e^{i(k\psi + lA)}$$

is a harmonic morphism with dilation given by

$$\lambda^2 = \sin^2 \beta \left(\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u} \right). \tag{3.15}$$

The metric on $\mathbb{S}^3$ is induced by the metric on $\mathbb{R}^4$. By taking derivatives along u, ψ and A, we get the following orthonormal basis of tangent vectors to $\mathbb{S}^3$:

$$\epsilon_1 = (-\sin u \cos \psi, -\sin u \sin \psi, \cos u \cos A, \cos u \sin A)$$

$$\epsilon_2 = (-\sin \psi, \cos \psi, 0, 0)$$

$$\epsilon_3 = (0, 0, -\sin A, \cos A)$$

Note that $\frac{\partial}{\partial u} = \epsilon_1$, $\frac{\partial}{\partial \psi} = \cos u \epsilon_2$, $\frac{\partial}{\partial A} = \sin u \epsilon_3$.
We compute $\frac{\partial \phi}{\partial \psi}$ and $\frac{\partial \phi}{\partial A}$ and we derive that

$$d\phi(l \cos u \epsilon_2 - k \sin u \epsilon_3) = 0$$

hence the horizontal space H in $\mathbb{S}^3$ w.r.t. ϕ consists in the vectors tangent to $\mathbb{S}^3$ and orthogonal to $V = l \cos u \epsilon_2 - k \sin u \epsilon_3$. It is generated by

$$H_1 = \frac{\partial}{\partial u} = \epsilon_1, \quad H_2 = k \sin u \epsilon_2 + l \cos u \epsilon_3$$

We compute in $\mathbb{R}^3$ that $< d\phi(H_1), d\phi(H_2) >= 0$. So the horizontal conformality of ϕ reduces to requiring that

$$\|d\phi(H_1)\|^2 = \|\frac{\partial \phi}{\partial u}\|^2 = \frac{\|d\phi(H_2)\|^2}{\|H_2\|^2} = \lambda^2$$

where λ is given by (3.15).

$$\frac{\partial \phi}{\partial u} = \left(\frac{\partial \beta}{\partial u}\right) (-\sin \beta, \cos \beta e^{i(k\psi + lA)}). \tag{3.16}$$

Then the condition for ϕ to be horizontally conformal is

$$\left(\frac{\partial \beta}{\partial u}\right)^2 = \sin^2 \beta \left(\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u} \right) \tag{3.17}$$

Case 1: $|k| = |l|$. Then Eq. (3.17) takes the form

$$\frac{1}{\sin^2 \beta}\left[\left(\frac{\partial \beta}{\partial u}\right)^2\right] = \frac{4k^2}{\sin^2 2u}, \tag{3.18}$$

which can be solved explicitly as follows. Set

$$v = \frac{\partial}{\partial u}.$$

We have that

$$\begin{aligned}
v\left(\log\tan\frac{\beta}{2}\right) &= \frac{\partial}{\partial \beta}\left(\log\tan\frac{\beta}{2}\right)v(\beta)\\
&= \frac{1}{\tan\frac{\beta}{2}}\frac{1}{2\cos^2\frac{\beta}{2}}v(\beta)\\
&= \frac{1}{2\sin\frac{\beta}{2}\cos\frac{\beta}{2}}v(\beta)\\
&= \frac{1}{\sin\beta}v(\beta)\\
&= \frac{1}{\sin\beta}\frac{\partial\beta}{\partial u}.
\end{aligned}$$

Then the left-hand side of Eq. (3.18) is equal to

$$v\left(\log\tan\frac{\beta}{2}\right)v\left(\log\tan\frac{\beta}{2}\right). \tag{3.19}$$

On the other hand we have that

$$\begin{aligned}
v\left(\log\tan^k u\right) &= v\left(k\log\tan u\right)\\
&= \frac{k}{\cos^2 u\tan u}\\
&= \frac{k}{\cos u\sin u}\\
&= \frac{2k}{\sin 2u}.
\end{aligned}$$

Then the right-hand side of Eq. (3.18) is equal to

$$v\left(\log\tan^k u\right)v\left(\log\tan^k u\right).$$

by the substitution in Eq. (3.18) we obtain

$$v\left(\log\tan\frac{\beta}{2}\right) = v\left(\log\tan^k u\right)$$

yielding the solution

$$\beta(u) = 2\arctan\left(\tan^k u\right) \tag{3.20}$$

Case 2: $|k| \neq |l|$. Now the reduction equation for horizontal conformality becomes

$$\frac{1}{\sin^2\beta}\left[\left(\frac{\partial\beta}{\partial u}\right)^2\right] = \frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}. \tag{3.21}$$

In order to proceed as before, we must write $\sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}}$ as a derivative .
We pose

$$\sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}} = \frac{\partial I}{\partial u}, \tag{3.22}$$

then we find an explicit formula for I. First we must evaluate the integral

$$I = \int \sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}}du.$$

There are two cases:
(a)$l^2 > k^2$.
We have

$$\sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}} = \frac{l}{\cos u \sin u}\sqrt{1 - \frac{l^2-k^2}{l^2}\sin^2 u}$$

First make the substitution:

$$\sin\theta = \frac{\sqrt{l^2-k^2}}{l}\sin u. \tag{3.23}$$

For the derivative we obtain:

$$\cos\theta d\theta = \frac{\sqrt{l^2-k^2}}{l}\cos u du. \tag{3.24}$$

Then

$$\sin^2\theta = \frac{l^2-k^2}{l^2}\sin^2 u$$

$$\cos^2\theta = 1 - \frac{l^2-k^2}{l^2}\sin^2 u$$

$$\cos\theta = \sqrt{1 - \frac{l^2-k^2}{l^2}\sin^2 u}$$

$$\cos^2 u = 1 - \frac{l^2}{l^2-k^2}\sin^2\theta = \frac{l^2-k^2-l^2\sin^2\theta}{l^2-k^2} \tag{3.25}$$

Then by using (3.23),(3.24) and (3.25) we obtain the integral

$$\begin{aligned} I &= \int \frac{l^2\cos^2\theta}{\cos^2 u \sin u\sqrt{l^2-k^2}}d\theta \\ &= \int \frac{l\cos^2\theta}{\cos^2 u \sin\theta}d\theta \\ &= \int \frac{l\cos^2\theta(l^2-k^2)}{\sin\theta(l^2-k^2-l^2\sin^2\theta)}d\theta. \end{aligned}$$

On the other hand we have the following equality

$$\frac{l}{\sin\theta} + \frac{lk^2\sin\theta}{l^2-k^2-l^2\sin^2\theta} = \frac{l\cos^2\theta(l^2-k^2)}{\sin\theta(l^2-k^2-l^2\sin^2\theta)}$$

then we obtain

$$I = l\int\frac{1}{\sin\theta}d\theta + lk^2\int\frac{\sin\theta}{l^2-k^2-l^2\sin^2\theta}d\theta.$$

The second of these integrals is easily evaluated after substituting $\phi = \cos\theta$ and we obtain

$$I = l\int\frac{-d\phi}{1-\phi^2} + lk^2\int\frac{-d\phi}{l^2\left(\phi^2-\left(\frac{k}{l}\right)^2\right)} = \frac{1}{2}l\log\left|\frac{1-\phi}{1+\phi}\right| + \frac{1}{2}k\log\left|\frac{k+l\phi}{k-l\phi}\right|.$$

Let

$$p(u) = \sqrt{l^2\cos^2 u + k^2\sin^2 u},$$

then

$$\begin{aligned} p(u)^2 &= l^2\cos^2 u + k^2\sin^2 u \\ &= (k^2-l^2)\sin^2 u + l^2 \\ &= -l^2\sin^2\theta + l^2 \\ &= l^2\cos^2\theta \end{aligned}$$

We thus obtain :

$$\left|\frac{k+p}{k-p}\right| = \left|\frac{k+l\cos\theta}{k-l\cos\theta}\right| = \left|\frac{k+l\phi}{k-l\phi}\right|$$

and

$$\left|\frac{l+p}{l-p}\right| = \left|\frac{1+\phi}{1-\phi}\right|.$$

Hence

$$\begin{aligned} I &= \frac{1}{2}l\log\left|\frac{l-p}{l+p}\right| + \frac{1}{2}k\log\left|\frac{k+p}{k-p}\right| \\ &= \log\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}} + \log\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}} \\ &= \log\left\{\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}}\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}}\right\}. \end{aligned}$$

By the substitution of the two side of the Eq. (3.21) and from (3.22) we obtain

$$v\left(\log\tan\frac{\beta}{2}\right) = \frac{\partial I}{\partial u} = v\left(\log\left\{\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}}\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}}\right\}\right)$$

yielding the solution

$$\beta(u) = 2\arctan\left\{\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}}\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}}\right\} \tag{3.26}$$

(b)$l^2 < k^2$.
Similarly, we suppose

$$\sinh\theta = \frac{\sqrt{k^2-l^2}}{l}\sin u$$

now involving hyperbolic functions, that gives us

$$\begin{aligned} I &= l\int \frac{1}{\sinh\theta}d\theta + lk^2\int \frac{\sinh\theta}{l^2-k^2-l^2\sinh^2\theta}d\theta \\ &= lI_1 + lk^2 I_2. \end{aligned}$$

It is easily evaluated after substituting $\phi = \cosh\theta$ and we obtain

$$\begin{aligned} I_1 &= \int \frac{d\phi}{\phi^2-1} \\ &= \int \frac{d\phi}{2(\phi-1)} - \int \frac{d\phi}{2(\phi+1)} \\ &= \frac{1}{2}\log|\phi-1| - \frac{1}{2}\log|\phi+1| \\ &= \frac{1}{2}\log\left|\frac{\phi-1}{\phi+1}\right| \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int \frac{d\phi}{l^2\left(\left(\frac{k}{l}\right)^2-\phi^2\right)} \\ &= \int \frac{1}{2lk}\log\left|\frac{k+l\phi}{k-l\phi}\right| \end{aligned}$$

then

$$I = \frac{1}{2}l\log\left|\frac{\phi-1}{\phi+1}\right| + \frac{1}{2}k\log\left|\frac{k+l\phi}{l\phi-k}\right|$$

Using the following two equalities

$$\frac{\phi-1}{\phi+1} = \frac{p-l}{p+l}$$

and

$$\frac{p+k}{p-k} = \frac{l\phi+k}{l\phi-k} \text{ with } p = l\phi,$$

we obtain

$$I = \log\left|\frac{p-l}{p+l}\right|^{\frac{l}{2}}\left|\frac{k+p}{p-k}\right|^{\frac{k}{2}}$$

By Eq. (3.21) we obtain

$$v\left(\log\tan\frac{\beta}{2}\right) = v\left(\log\left|\frac{p-l}{p+l}\right|^{\frac{l}{2}}\left|\frac{k+p}{p-k}\right|^{\frac{k}{2}}\right)$$

yielding the solution

$$\beta(u) = 2\arctan\left\{\left|\frac{p-l}{p+l}\right|^{\frac{l}{2}}\left|\frac{k+p}{p-k}\right|^{\frac{k}{2}}\right\}. \tag{3.27}$$

3.5 The Preimages of $\Phi_{k,l}$

In this section, we take a point P in $\mathbb{S}^2$ and we look for its preimage in $\mathbb{S}^4$ by $\Phi_{k,l}$. First, we look for the preimage of this point in $\mathbb{S}^3$ by the map $\phi_{k,l}$ and then we fix a point on this preimage and look for its preimage in $\mathbb{S}^4$ by the map F.

3.5.1 The preimage of F

We recall definition of the map F in (3.7)
$F : \mathbb{S}^4 \to \mathbb{S}^3$ for $s \in [0, \pi]$, $t \in [0, \pi/2]$ and $a, b \in [0, 2\pi]$ and

$$F\left(\cos s, \sin s\ \left(\cos t\ e^{ia},\ \sin t\ e^{ib}\right)\right) = \left(\cos\alpha(s), \sin\alpha(s)\cos 2t, \sin\alpha(s)\sin 2t\ e^{i(a+b)}\right).$$

where α is a increasing regular function such that $\alpha(0) = 0$ and $\alpha(\pi) = \pi$

Proposition 9. *Let $P \in \mathbb{S}^3$,*

1) If $P \neq (1, 0, 0, 0)$, then $F^{-1}(P)$ is a closed loop.

2) $F^{-1}(\pm 1, 0, 0, 0) = \{(\pm 1, 0, 0, 0, 0)\}$

Proof. We fix $Z \in \mathbb{S}^2$ and let $P = (\cos\alpha_0, \sin\alpha_0 Z)$ with $Z \in \mathbb{S}^2$ and $\alpha_0 \in [0, \pi]$. Now we look for its preimage in $\mathbb{S}^4$. There exists a unique s_0 such that $\alpha_0 = \alpha(s_0)$.

1) If $\sin\alpha_0 \neq 0$,
$$F^{-1}(\pi) = \{(\cos s_0, \sin s_0 x) : H(x) = Z\} \tag{3.28}$$

2) If $\sin\alpha_0 = 0$, then $P = (\pm 1, 0, 0, 0)$. Moreover if $\alpha_0 = 0$ (resp. $\alpha_0 = \pi$) then $s_0 = 0$ (resp. $s_0 = \pi$) and 2) follows. □

3.5.2 The preimage of $\phi_{k,l}$

We denote by $N_{\mathbb{S}^2}$ (resp $S_{\mathbb{S}^2}$) the north pole $(1, 0, 0)$ (resp. south pole $(-1, 0, 0)$).
We also recall definition of $\phi_{k,l} : \mathbb{S}^3 \to \mathbb{S}^2$ given in (3.11) :

$$\phi_{k,l}\left(\cos u\ e^{i\psi}, \sin u\ e^{i(a+b)}\right) = \left(\cos\beta(u), \sin\beta(u)\ e^{i(k\psi + l(a+b))}\right) \tag{3.29}$$

Proposition 10. *The map $\phi_{1,1}$ is the Hopf map so $\phi_{1,1}^{-1}(Q)$ is a great circle in $\mathbb{S}^3$. More generally, $\phi_{k,l}^{-1}(\{Q\})$ is a (k, l) torus-knot if $Q \neq N_{\mathbb{S}^2}, S_{\mathbb{S}^2}$ and $\phi_{k,l}^{-1}(\{N_{\mathbb{S}^2}\})$ and $\phi_{k,l}^{-1}(\{S_{\mathbb{S}^2}\})$ are great circles in $\mathbb{S}^3$.*

Proof. Let $Q = (\cos v_0, \sin v_0\ e^{i\mu_0})$ with $v_0 \in [0, \pi]$ and $\mu_0 \in [0, 2\pi]$.
There exists a unique $u_0 \in \left[0, \frac{\pi}{2}\right]$ such that $v_0 = \beta(u_0)$.
If $v_0 = 0$ (resp. $v_0 = \pi$) i.e. $Q = N_{\mathbb{S}^2}$ (resp. $Q = S_{\mathbb{S}^2}$), then $u_0 = 0$ (resp. $u_0 = \frac{\pi}{2}$) and $\phi_{k,l}^{-1}(\{N_{\mathbb{S}^2}\}) = \{(e^{i\Psi}, 0) : \Psi \in [0, 2\pi]\}$,
resp. $\phi_{k,l}^{-1}(\{S_{\mathbb{S}^2}\}) = \{(0, e^{iA}) : A \in [0, 2\pi]\}$.

Now assume $Q = (\cos v_0, \sin v_0\ e^{i\mu_0})$ with $v_0 \in]0, \pi[$ and $\mu_0 \in [0, 2\pi]$.
The preimage of Q is

$$\phi_{k,l}^{-1}(Q) = \left\{(\cos u_0\ e^{i\psi}, \sin u_0 e^{iA}) : \mu_0 = k\psi + lA\ \Psi, A \in [0, 2\pi]\right\}.$$

We obtain a torus knot of type (k, l); it is included in the torus on $\mathbb{S}^3$ given by

$$T_{u_0} = \left\{(\cos(u_0)\ e^{i\psi}, \sin(u_0)\ e^{iA}) : \psi \in [0, 2\pi] \text{ and } A \in [0, 2\pi]\right\}.$$

$\square$

3.5.3 The preimage of the North pole $N_{\mathbb{S}^2} = (1, 0, 0)$ of $\mathbb{S}^2$ by $\Phi_{k,l}$

In this section, we find the preimage of the North pole $N_{\mathbb{S}^2} = (1, 0, 0)$ by the map $\Phi_{k,l}$. We also recall the definition of $\Phi_{k,l} = \phi_{k,l} \circ F$: where

$$\phi_{k,l}\left(\cos u\ e^{i\psi}, \sin u\ e^{i(a+b)}\right) = \left(\cos\beta(u), \sin\beta(u)\ e^{i(k\psi+l(a+b))}\right) \tag{3.30}$$

and

$$F\left(\cos s, \sin s\ \left(\cos t\ e^{ia},\ \sin t\ e^{ib}\right)\right) = \left(\cos\alpha(s), \sin\alpha(s)\cos 2t \sin\alpha(s)\sin 2t\ e^{i(a+b)}\right).$$

Proposition 11. *The preimage of the north pole $N_{\mathbb{S}^2} = (1, 0, 0)$ in $\mathbb{S}^2$ by the map $\Phi_{k,l}$ is the union of the two totally geodesic 2-spheres*

$$S_1 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_4 = x_5 = 0\}$$

and

$$S_2 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = 0\},$$

with $S_1, S_2 \subset \mathbb{S}^4 \subset \mathbb{R}^5$.
The spheres S_1 and S_2 intersect at each pole $N_{\mathbb{S}^4} = (1, 0, 0, 0, 0)$ and $S_{\mathbb{S}^4} = (-1, 0, 0, 0, 0)$ with opposite signs of intersection.

Proof. We look for a point of the form $\left(\cos s, \sin s \cos t e^{ia}, \sin s \cos t e^{ib}\right) \in \mathbb{S}^4$.

Let $Q = \left(\cos u e^{i\psi}, \sin u e^{iA}\right) \in \mathbb{S}^3$ with $\phi_{k,l}(Q) = N_{\mathbb{S}^2}$. Then $\beta(u) = 0$ hence $u = 0$. The preimage of $N_{\mathbb{S}^2}$ in $\mathbb{S}^3$ is given for $u = 0$ by $\{(e^{i\psi}, 0)\} \in \mathbb{S}^3 \subset \mathbb{C}^2$.

We fix ψ and we look for the preimage of $(e^{i\psi}, 0)$ in $\mathbb{S}^4$.
Looking at the two equations (4.2) and (3.9), we obtain by a small calculation the following

$$\sin\alpha(s)\sin 2t = 0. \tag{3.31}$$

$$\cos\alpha(s) + i\sin\alpha(s)\cos 2t = \ e^{i\psi(s,t)} \tag{3.32}$$

Then, $\sin\alpha(s) = 0$ or $\sin 2t = 0$.

1) If $\sin\alpha(s) = 0$ then $\sin s = 0$ therefore $s = 0$ or $s = \pi$. Using (3.32), we have $e^{i\psi(s,t)} = \pm 1$ then, $\psi = 0$ or $\psi = \pi$. Here, we obtain the two poles $N_{\mathbb{S}^4} = (1,0,0,0,0)$ and $S_{\mathbb{S}^4} = (-1,0,0,0,0)$.

2) On the other hand, if $\sin 2t = 0$ then $t = 0$ or $t = \dfrac{\pi}{2}$. Using (3.32), we obtain for $\psi \neq 0$ and $\psi \neq \pi$ two cases :

   a) If $0 < \psi < \pi$, then $\alpha = \psi$ and $t = 0$, then, we obtain in $\mathbb{S}^4$, $(\cos s, \sin s(e^{ia}, 0))$ where $a \in [0, 2\pi]$ and $s \in]0, \pi[$.
   Here we have the sphere S_1 punctured at the two poles.

   b) If $\pi < \psi < 2\pi$, then $\alpha = 2\pi - \psi$ and $t = \dfrac{\pi}{2}$ therefore we obtain in $\mathbb{S}^4$, $(\cos s, \sin s(0, e^{ib}))$ where $b \in [0, 2\pi]$ and $s \in]0, \pi[$.
   Here we have the sphere S_2 punctured at the two poles.

In case one we obtain the two poles $N_{\mathbb{S}^4}$ and $S_{\mathbb{S}^4}$.
In case two we obtain the two great spheres S_1 and S_2 minus the poles $N_{\mathbb{S}^4}$ and $S_{\mathbb{S}^4}$.
Putting cases 1. and 2. together shows that the preimage of $N_{\mathbb{S}^2}$ consists of two 2-spheres S_1 and S_2 intersecting transversally at the poles $N_{\mathbb{S}^4}$ and $S_{\mathbb{S}^4}$.
Since, $H_2(\mathbb{S}^4, \mathbb{Z}) = 0$, S_1 and S_2 have a zero total number of intersection points (counted with sign). Hence, $N_{\mathbb{S}^4}$ and $S_{\mathbb{S}^4}$ are intersection points of opposite signs.

□

In fact we can check by hand that the two intersection points have different signs. For that we choose a positive orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathbb{R}^5$ where $e_1 = N_{\mathbb{S}^4}$. We can see clearly that S_1 and S_2 are the intersection of $\mathbb{S}^4$ with the two subspaces of $\mathbb{R}^5$ generated by $\{e_1, e_2, e_3\}$ (resp. $\{e_1, e_4, e_5\}$).
Let $N_{\mathbb{S}^4} = (1,0,0,0,0)$ and $S_{\mathbb{S}^4} = (-1,0,0,0,0)$ be the two intersection points of S_1 and S_2.
First, for $N_{\mathbb{S}^4} \in S_1 \cap S_2$ we have :

- $T_N S_1$ and $T_N S_2$ are generated by the two positive bases $\{e_2, e_3\}$ resp. $\{e_4, e_5\}$ and $T_N \mathbb{S}^4$ is generated by the positive basis $\{e_2, e_3, e_4, e_5\}$. So the orientation at N is positive.

Now, we take $S_{\mathbb{S}^4} \in S_1 \cap S_2$ we have :

- $T_P S_1$ and $T_P S_2$ are generated by the two positive bases $\{-e_2, e_3\}$ resp. $\{-e_4, e_5\}$ and $T_P \mathbb{S}^4$ is generated by the positive basis $\{-e_2, e_3, e_4, e_5\}$. So the orientation at this point P is negative.

Therefore, the two intersection points have opposite signs.

3.5.4 The preimage of the South pole $S_{\mathbb{S}^2} = (-1,0,0)$ in $\mathbb{S}^2$ by $\Phi_{k,l}$

In this section we look for the preimage of the second pole $S_{\mathbb{S}^2} = (-1,0,0)$ by the map $\Phi_{k,l}$.

Proposition 12. *The preimage of the pole south $S_{\mathbb{S}^2} = (-1,0,0)$ in $\mathbb{S}^2$ is a Clifford torus in the equator of $\mathbb{S}^4$.*

Proof. If $\beta(u) = \pi$, we have

$$\cos\beta(u) = -1 \text{ and } u = \pi/2.$$

The preimage of this pole in $\mathbb{S}^3$, is given for $u = \dfrac{\pi}{2}$ by

$$\{(0, e^{iA})\} \in \mathbb{S}^3 \subset \mathbb{C}^2.$$

We fix A and we look for the preimage of $(0, e^{iA})$ in $\mathbb{S}^4$. Looking at (3.8) and (3.9), as above we get the two equations:

$$\cos\alpha(s) + i\sin\alpha(s)\cos 2t = 0 \tag{3.33}$$

$$\sin\alpha(s)\sin 2t = 1 \tag{3.34}$$

Therefore,

$$\alpha(s) = \pi/2 \text{ and } 2t = \pi/4.$$

By a small computation $\alpha\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$; since α is strictly increasing we conclude that $s = \frac{\pi}{4}$. We conclude that the preimage in $\mathbb{S}^4$ of the south pole $(-1,0,0)$ is a Clifford torus T in the equator $\mathbb{S}^3$ of $\mathbb{S}^4$:

$$T := \left\{ \left(0, \frac{\sqrt{2}}{2}\, e^{ia}, \frac{\sqrt{2}}{2}\, e^{ib}\right) : (a,b) \in [0,2\pi] \times [0,2\pi] \right\}$$

□

3.6 Critical points of $\Phi_{k,l}$

In this section we are going to find the critical points of the map $\Phi_{k,l}$. To do this, we need to prove the following theorem:

Theorem 10. *The set of critical points of $\Phi_{k,l}$ for $k = l = 1$ is given by two 2-spheres having the two poles as intersection points. Otherwise, if $k, l \neq 1$ the set of critical points are the preimages of the north pole (the same two spheres as for $k = l = 1$) together with the preimage of the south pole (a torus).*

3.6.1 Critical points of F

We investigate the map F from $\mathbb{S}^4$ into $\mathbb{S}^3$ given by (3.7). For $0 < s < \pi$, $\alpha'(s) \neq 0$. It follows that all points of $\mathbb{S}^4$ are regular for F, outside of the poles. We now investigate what happens at the North and South poles.

We look at a neighbourhood of the pole $N_{\mathbb{S}^4} = (1,0,0,0,0)$. Near the pole $N_{\mathbb{S}^4} =$

$(1,0,0,0,0)$, the parameter s is close to 0 so we identify a neighborhood of $N_{\mathbb{S}^4}$ with a 4-ball centred at $N_{\mathbb{S}^4}$.

$$\mathfrak{B}_\epsilon = \left\{ sx : (s,x) \in [0,\epsilon] \times \mathbb{S}^3 \right\},$$

By projection on the last two coordinates we identify a neighborhood of the north pole $N_{\mathbb{S}^2}$ in $\mathbb{S}^2$ to a disc D of $\mathbb{R}^2$.
Now consider the regular function

$$\alpha(s) = 2 \arctan \left(\tan^2 \left(\frac{s}{2} \right) \right).$$

For $s \sim 0$, we have

$$\alpha(s) \sim 2 \arctan \frac{s^2}{4}.$$

Consequently,

$$\alpha(s) \sim \frac{s^2}{2}.$$

Hence

$$(\cos \alpha(s), \sin \alpha(s) H(x)) \sim \left(1 - \frac{s^4}{4}, \frac{s^2}{2} H(x) \right). \tag{3.35}$$

Under the above identifications we write F as

$$sx \longrightarrow \frac{s^2}{2} H(x).$$

It follows that the North pole $N_{\mathbb{S}^4}$ is a critical point for F.

In the second step we look at a neighbourhood of the pole $S_{\mathbb{S}^4} = (-1,0,0,0,0)$, here we are going to use the same procedure that we use for the other pole.
So we identify a neighborhood of S with a 4-ball centred at $S_{\mathbb{S}^4}$.
Near the pole $S_{\mathbb{S}^4} = (-1,0,0,0,0)$, the parameter s is close to π, for $s \sim \pi$, we put $s' = \pi - s$, now $s' \sim 0$, then

$$\sin s = \sin(\pi - s') = \sin s' \sim s' = \pi - s.$$

For a small $\epsilon > 0$, the set $\left\{ (\pi - s)x : (\pi - s, x) \in [0,\epsilon] \times \mathbb{S}^3 \right\}$ parametrizes a neighborhood of the south pole $S_{\mathbb{S}^4}$.
The function

$$\alpha(s) = \alpha(\pi - s') \sim 2 \arctan \frac{4}{s'^2}(1 + o(s')) \sim \frac{4}{s'^2}(1 + o(s')) \sim \frac{4}{(\pi - s)^2}(1 + o(\pi - s))$$

or,

$$\alpha(s) \sim \frac{4}{(\pi - s)^2}(1 + o(\pi - s)).$$

We can write F in this neighborhood as

$$(\cos \alpha(s), \sin \alpha(s) H(x)) \sim \left((1 + o(\pi - s), \frac{(s - \pi)^2}{2} H(x) \right). \tag{3.36}$$

Under the above identification we write F as

$$s'x \longrightarrow \frac{s'^2}{2} H(x).$$

It's clear that the south pole is a critical point of the map F.

3.6.2 Estimate of β near the endpoints of $\left[0, \frac{\pi}{2}\right]$

We recall that $\beta : \left[0, \frac{\pi}{2}\right] \to [0, \pi]$ is a regular function of u such that $\beta(0) = 0$ and $\beta\left(\frac{\pi}{2}\right) = \pi$.
Given by the formula for $k \neq l$

$$\beta(u) = 2\arctan\left\{\left|\frac{l - p(u)}{l + p(u)}\right|^{\frac{l}{2}} \left|\frac{k + p(u)}{k - p(u)}\right|^{\frac{k}{2}}\right\}, \tag{3.37}$$

with

$$p(u) = \sqrt{l^2 \cos^2 u + k^2 \sin^2 u}. \tag{3.38}$$

For $k = l$ the formula is

$$\beta(u) = 2\arctan\left(\tan^k u\right). \tag{3.39}$$

Lemma 13. *Let $\beta : \left[0, \frac{\pi}{2}\right] \to [0, \pi]$ as above*

1) For $u \sim 0$ we have

$$\beta(u) = Cu^l + o(u^l) \text{ with } C \in \mathbb{R}^+. \tag{3.40}$$

2) For $u \sim \frac{\pi}{2}$, let $v = u - \frac{\pi}{2}$ we have

$$\beta(u) = \pi - C\left(u - \frac{\pi}{2}\right)^k + o(v^k) \text{ with } C \in \mathbb{R}^+. \tag{3.41}$$

Proof.

1) We shall examine its behavior near a critical point, for this we use Taylor's Formula. We have

$$p^2 = l^2 \cos^2 u + k^2 \sin^2 u.$$

Then, in a neighborhood of 0 we have :

$$\begin{aligned}
p^2 &= l^2\left(1 - \frac{u^2}{2}\right)^2 + k^2u^2 + o(u^2) \\
&= l^2 + (k^2 - l^2)u^2 + o(u^2) \\
&= l^2\left(1 + \frac{k^2 - l^2}{l^2}u^2\right) + o(u^2)
\end{aligned}$$

Then,

$$\begin{aligned}
p &= l\sqrt{1 + \frac{k^2 - l^2}{l^2}u^2} + o(u^2) \\
&= l\left(1 + \frac{k^2 - l^2}{2l^2}u^2\right) + o(u^2)
\end{aligned}$$

We derive

$$\begin{aligned}
l-p &= \frac{l^2-k^2}{2l}u^2+o(u^2)\\
l+p &= 2l+\frac{k^2-l^2}{2l}u^2+o(u^2)\\
\frac{l-p}{l+p} &= \frac{l^2-k^2}{4l^2}u^2+o(u^2)
\end{aligned}$$

and

$$\begin{aligned}
k-p &= k-l-\frac{k^2-l^2}{2l}u^2+o(u^2)\\
k+p &= l+k+\frac{k^2-l^2}{2l}u^2+o(u^2)\\
\frac{k+p}{k-p} &= \frac{k+l}{k-l}+o(u).
\end{aligned}$$

So, we get

$$\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}}=\left|\frac{l^2-k^2}{4l^2}\right|^{\frac{l}{2}}u^l=C_1u^l+o(u^l) \tag{3.42}$$

and

$$\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}}=\left|\frac{k+l}{k-l}\right|^{\frac{k}{2}}=C_2+o(u). \tag{3.43}$$

We put $C_3=C_1C_2$, then the product of the two estimates above (3.42) and (3.43) gives us the following

$$\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}}\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}}=C_3u^l+o(u^l). \tag{3.44}$$

Finally, using (3.44) we obtain for $\beta(u)$ the following

$$\begin{aligned}
\beta(u) &= 2\arctan(C_3u^l)+o(u^l)\\
&= 2C_3u^l+o(u^l)
\end{aligned}$$

Consequently,

$$\beta(u)=Cu^l+o(u^l) \text{ with } C=2C_3. \tag{3.45}$$

2) Now we are going to use the same procedure as in the proof of 1) but this time in a neighborhood of $\frac{\pi}{2}$. Let $v=\frac{\pi}{2}-u\geqslant 0$. Then, using the trigonometric formulas, we obtain the following

$$\begin{aligned}
p^2 &= l^2\cos^2 u+k^2\sin^2 u\\
&= l^2\cos^2(\frac{\pi}{2}-v)+k^2\sin^2(\frac{\pi}{2}-v)\\
&= l^2\sin^2 v+k^2\cos^2 v.
\end{aligned} \tag{3.46}$$

Then, for v in a neighborhood of 0, we have

$$\begin{aligned} p^2 &= l^2v^2 + k^2\left(1 - \frac{v^2}{2}\right)^2 + o(v^2) \\ &= k^2 + (l^2 - k^2)v^2 + o(v^2) \\ &= k^2\left(1 + \frac{l^2 - k^2}{k^2}v^2\right) + o(v^2). \end{aligned}$$

Then,

$$\begin{aligned} p &= k\sqrt{1 + \frac{l^2 - k^2}{k^2}v^2} + o(v^2) \\ &= k\left(1 + \frac{l^2 - k^2}{2k^2}v^2\right) + o(v^2) \end{aligned}$$

We derive,

$$\begin{aligned} k - p &= \frac{k^2 - l^2}{2k}v^2 + o(v^2) \\ k + p &= 2k + \frac{l^2 - k^2}{2k}v^2 + o(v^2) \\ \frac{k+p}{k-p} &= \frac{4k^2}{k^2 - l^2}\frac{1}{v^2}\left(1 + o(v)\right) \end{aligned}$$

and

$$\begin{aligned} l - p &= l - k + \frac{k^2 - l^2}{2k}v^2 + o(v^2) \\ l + p &= l + k + \frac{l^2 - k^2}{2k}v^2 + o(v^2) \\ \frac{l-p}{l+p} &= \frac{l-k}{l+k} + o(v^2). \end{aligned}$$

So, we get

$$\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}} = \left|\frac{l-k}{l+k}\right|^{\frac{l}{2}} = C_1 + o(v^2) \tag{3.47}$$

and

$$\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}} = \left|\frac{4k^2}{l^2 - k^2}\right|^{\frac{k}{2}}\frac{1}{v^{2k}}\left(1 + o(v)\right) = \frac{C_2(1 + o(v))}{v^{2k}}. \tag{3.48}$$

We put $C_3 = C_1C_2$, then the product of the two estimates above (3.47) and (3.48) gives us the following

$$\left|\frac{l-p}{l+p}\right|^{\frac{l}{2}}\left|\frac{k+p}{k-p}\right|^{\frac{k}{2}} = \frac{C_3\left(1 + o\left(v\right)\right)}{v^k}. \tag{3.49}$$

We let $y = \frac{\beta(u)}{2}$ be such that

$$LHS(3.49) = \tan y.$$

We put $z = y - \dfrac{\pi}{2}$, then

$$\frac{\cos y}{\sin y} \sim -z\frac{1}{1-\frac{z^2}{2}} \sim -z + o(z^2).$$

Therefore,

$$\beta(v) \sim \pi - 2C_3 v^k. \tag{3.50}$$

Finally, using (3.50) we obtain for $\beta(u)$ the following

$$\beta(u) = \pi - C\left(\frac{\pi}{2} - u\right)^k \text{ with } C = 2C_3. \tag{3.51}$$

□

3.6.3 Critical points of $\phi_{k,l}$

We go back to the map from $\mathbb{S}^3$ into $\mathbb{S}^2$ given by

$$\phi_{k,l}(\cos ue^{i\psi}, \sin ue^{iA}) = (\cos\beta(u), \sin\beta(u)e^{i(k\psi+lA)}),$$

where $u \in \left[0, \frac{\pi}{2}\right]$ and $\psi, A \in [0, 2\pi]$.
We know that the only critical values of $\phi_{k,l}$ are the south and north poles $S_{\mathbb{S}^2}, N_{\mathbb{S}^2}$.

1) The North pole $N_{\mathbb{S}^2} = (1,0,0)$: for $u = 0$, the corresponding points of $\mathbb{S}^3$ are of the form $P = (e^{i\psi_0}, 0, 0)$, with $\phi_{k,l}(P) = (1,0,0)$. We now investigate the behaviour of $\phi_{k,l}$ in a neighborhood of such a P.
We take $\eta > 0$ small. We identify a neighborhood of the point $P \in \mathbb{S}^3$ with $[0,\eta] \times [0,2\pi] \times [0,2\pi]$ by setting

$$\left(u, e^{i\psi}, e^{iA}\right) \longmapsto \left(\cos ue^{i\psi}, \sin ue^{iA}\right). \tag{3.52}$$

Note that

$$(u, A) \longmapsto z = \sin ue^{iA}, \tag{3.53}$$

parametrizes a disk in polar coordinates for $0 \leqslant \sin u \leqslant \eta$ and $0 \leqslant A \leqslant 2\pi$.
We write the point of $\mathbb{S}^2$ as $\left(\cos v, \sin ve^{i\mu}\right)$ and we identify a neighborhood of $N_{\mathbb{S}^2} = (1,0,0)$ with the disk $\{\sin ve^{i\mu} : v \in [0,\eta[\text{ and } e^{i\mu} \in \mathbb{S}^1\}$.

Lemma 14. *In these two coordinates systems $\phi_{k,l}$ can be written in a neighborhood of the North pole $N_{\mathbb{S}^2}$ as*

$$(e^{i\psi}, z) \longmapsto Cz^l e^{ik\psi} \tag{3.54}$$

Proof. For $u \sim 0$, the function

$$\beta(u) \sim Cu^l.$$

Then,

$$\sin\beta(u) \sim Cu^l.$$

Now $|z| = \sin u$, then

$$\sin\beta(u) \sim C|z|^l.$$

Let $e^{i\psi} \in \mathbb{S}^1$, then we have

$$\sin\beta(u)e^{i(k\psi+lA)} \sim C|z|^l e^{ilA} e^{ik\psi}.$$

Therefore, $\phi_{k,l}$ can be written as

$$\phi_{k,l} : (e^{i\psi}, z) \longmapsto Cz^l e^{ik\psi}.$$

□

2) The South pole $S_{\mathbb{S}^2} = (-1, 0, 0)$: now for $u = \dfrac{\pi}{2}$ we get points $Q \in \mathbb{S}^3$ of the form $Q = (0, 0, e^{iA})$, with $\phi_{k,l}(Q) = (-1, 0, 0)$. We now investigate the behavior of $\phi_{k,l}$ in a neighborhood of such a Q. We proceed as above.
We identify a neighborhood of Q with $[0, \eta] \times \mathbb{S}^1 \times [0, 2\pi]$ by setting

$$\left(u, e^{iA}, e^{i\psi}\right) \longmapsto \left(\cos u e^{i\psi}, \sin u e^{iA}\right). \tag{3.55}$$

Note that

$$(u, \psi) \longmapsto z = \cos u e^{i\psi}, \tag{3.56}$$

parametrizes a disk in polar coordinates for $0 \leqslant \cos u \leqslant \eta$ and $0 \leqslant \psi \leqslant 2\pi$.
We write the point of $\mathbb{S}^2$ as $\left(\cos v, \sin v e^{i\mu}\right)$ and we identify a neighborhood of $S = (-1, 0, 0)$ with the disk $\{\sin v e^{i\mu}$ with $v \in [0, \eta[$ and $e^{i\mu} \in \mathbb{S}^1\}$.

Lemma 15. *In these two coordinates system $\phi_{k,l}$ can be written in a neighborhood of the South pole $S_{\mathbb{S}^2}$ as*

$$(e^{iA}, z) \longmapsto Cz^k e^{ilA} \tag{3.57}$$

Proof. For $u \sim \frac{\pi}{2}$, we have

$$\beta(u) \sim \pi - C\left(\frac{\pi}{2} - u\right)^k.$$

Then,

$$\sin\beta(u) \sim C(\frac{\pi}{2} - u)^k.$$

Now $|z| = \cos u \sim \frac{\pi}{2} - u$, then

$$\sin\beta(u) \sim C|z|^k.$$

Let $e^{iA} \in \mathbb{S}^1$, then we have

$$\sin\beta(u)e^{i(k\psi+lA)} \sim C|z|^k e^{ik\psi} e^{ilA}.$$

Therefore, $\phi_{k,l}$ can be written as

$$\phi_{k,l} : (e^{iA}, z) \longmapsto Cz^k e^{ilA}.$$

□

3.7 Multiple fibres

3.7.1 Smooth multiple fibres

We begin by defining a notion of multiple fibres for harmonic morphisms.

Definition 11. *Let $\phi : M^m \longrightarrow N^n$ be a harmonic morphism and let p_0 be a critical value of ϕ in N^n such that $\Sigma = phi^{-1}(p_0)$ is smooth, connected and closed. The fibre $\Sigma = \phi^{-1}(p_0)$ is a multiple fibre of multiplicity μ if there exists*
1) a neighbourhood U of p_0 in N^n
2) a tubular neighbourhood $\mathcal{T}$ of Σ and a projection $\pi : \mathcal{T} \longrightarrow \Sigma$ such that
i) $\phi^{-1}(U) \subset \mathcal{T}$
ii) for every $p \in U$, $\phi^{-1}(p)$ is connected and compact
iii) for every $X \in \Sigma$ and every $p \in U$, $\pi^{-1}(X)$ and $\phi^{-1}(p)$ meet at μ points and these intersection points have the same sign.

We let $[\Sigma]$ be the homology class of Σ in $H_{m-n}(\mathcal{T}, \mathbb{Z})$. Then for p close enough to p_0, and by iii), the homology class of $[\phi^{-1}(p)]$ verifies

$$[\phi^{-1}(p)] = \pm\mu[\Sigma] \tag{3.58}$$

3.8 Multiple fibres of $\phi_{kl} : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$

Proposition 13. *We consider the harmonic morphism $\phi_{kl} : (\mathbb{S}^3, g_{kl}) \longrightarrow \mathbb{S}^2$.*
1) The preimage of $N_{\mathbb{S}^2} = (1, 0, 0)$ is a multiple fibre of multiplicity l.
2) The preimage of $S_{\mathbb{S}^2} = (-1, 0, 0)$ is a multiple fibre of multiplicity k

Proof. We write the proof for $N_{\mathbb{S}^2}$ and the proof for $S_{\mathbb{S}^2}$ is identical. We let $\Sigma = \phi_{kl}^{-1}(N_{\mathbb{S}^2}) = \{(e^{i\psi}, 0) \in \mathbb{S}^3\}$. We define a tubular neighbourhood $\mathcal{T}$ by

$$\mathcal{T} = \{(\cos u e^{i\psi}, \sin u e^{iA}) : \Psi, A \in [0, 2\pi],\ 0 \leqslant \sin u < \eta\} \tag{3.59}$$

We identify

$$\mathcal{T} \simeq \Sigma \times \mathbb{D}_\eta = \{(e^{i\psi}, z)\} \tag{3.60}$$

where

$$z = \sin u e^{iA} \in \mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\} \tag{3.61}$$

and the projection π becomes

$$(e^{i\psi}, z) \mapsto z \tag{3.62}$$

We identify a neighbourhood of $N_{\mathbb{S}^2}$ in $\mathbb{S}^2$ with a disk D in $\mathbb{C}$; under this identification, $N_{\mathbb{S}^2}$ is identified to 0. In the above identification of $\mathcal{T}$, we write

$$\phi_{kl} : \mathcal{T} \longrightarrow D \tag{3.63}$$

$$\phi_{kl} : (e^{i\psi}, z) \mapsto C(z) z^l e^{ik\psi} \tag{3.64}$$

where $C(z)$ is of the form $C(z) = C + o_1(|z|)$, C being a non-zero complex number. Now let $w \in D$. Using (3.64), we can write

$$\phi_{kl}^{-1}(w) = \{(e^{i\psi}, z) : C(z)z^l e^{ik\psi} = w\}$$

If $(e^{i\psi_0}, 0) \in \Sigma$, a point $(e^{i\psi_0}, z)$ belongs to $\pi^{-1}(e^{i\psi_0}, 0) \cap \phi_{kl}^{-1}(w)$ if

$$C(z)z^l e^{ik\psi_0} = w \tag{3.65}$$

We derive

Lemma 16. *The equation (3.65) has l preimages.*

We now prove

Lemma 17. *The intersections of $\phi_{kl}^{-1}(w)$ with the disk $\pi^{-1}(e^{i\psi_0}, 0)$ all have the same signs.*

Proof. We write the coordinate z in D as $z = x + iy$ and we compute the derivative $d\phi_{kl}$ on D; it verifies

$$\frac{\partial \phi_{kl}}{\partial x} = Clz^{l-1}e^{ik\psi_0} + o(|z|^l) \tag{3.66}$$

$$\frac{\partial \phi_{kl}}{\partial y} = i(Clz^{l-1}e^{ik\psi_0} + o(|z|^l)) \tag{3.67}$$

It follows from (3.66) and (3.67) that $Ker(d\phi_{kl})$ does not contain vectors tangent to a fibre of the tubular neighbourhood $\mathcal{T}$. Hence $\phi_{kl}^{-1}(w)$ is always transverse to the fibres of the tubular neighbourhood $\mathcal{T}$: since $\mathcal{T} \setminus \phi_{k,l}^{-1}(0)$ is connected, the sign of the intersections of $\phi_{kl}^{-1}(w)$ with one of the fibres of π will be of the same sign. $\square$

3.9 Singular multiple fibres

We need to adapt Def.11 to fit the case of a singular multiple fibre. First, we replace the tubular neighbourhood by the following object:

Definition 12. *Let $\phi : M^m \longrightarrow N^n$ be a harmonic morphism and let p_0 be a critical value of ϕ in N^n; suppose that $\Sigma = \phi^{-1}(p_0)$ is smooth except at a singular set $\mathcal{S}$ of codimension at least 2. An open set $\mathcal{T}$ together with a surjective map*

$$\pi : \mathcal{T} \longrightarrow \Sigma \tag{3.68}$$

is called a singular tubular neighbourhood of Σ if there exists a sequence of open neighbourhoods U_n of $\mathcal{S}$ such that the following is true:
i) $\mathcal{S} = \bigcap U_n$
ii) for every n, the restriction of π to $\pi^{-1}(\Sigma \backslash (\Sigma \cap U_n))$ is a tubular neighbourhood of $\Sigma \backslash (\Sigma \cap U_n)$.

We now give a modified version of Def.11.

Definition 13. *Let $\phi : M^m \longrightarrow N^n$ be a harmonic morphism and let p_0 be a critical value of ϕ in N^n such that $\Sigma = \phi^{-1}(p_0)$ is compact and smooth outside of a subset $\mathcal{S}$ of codimension at least 2.*
The fibre $\Sigma = \phi^{-1}(p_0)$ is a multiple fibre of multiplicity μ if there exists a singular tubular neighbourhood $\pi : \mathcal{T} \longrightarrow \Sigma$ of Σ such that for every $X \in \Sigma \backslash \mathcal{S}$, there exists a neighbourhood V_X of p_0 such that for every $p \in V_X$, $\pi^{-1}(X)$ and $\phi^{-1}(p)$ meet at μ points and these intersection points all have the same sign.

3.10 Multiple fibres of $\Phi_{kl} : \mathbb{S}^4 \longrightarrow \mathbb{S}^2$

Proposition 14. *We consider the harmonic morphism $\Phi_{kl} : (\mathbb{S}^4, g_{kl}) \longrightarrow \mathbb{S}^2$.*
1) The preimage of $S_{\mathbb{S}^2} = (-1, 0, 0)$ is a multiple fibre of multiplicity l
2) The preimage of $N_{\mathbb{S}^2} = (1, 0, 0)$ is a multiple fibre of multiplicity k.

Proof. Since the preimage of $S_{\mathbb{S}^2}$ is smooth and the preimage of $N_{\mathbb{S}^2}$ is not, we treat both cases separately.
1) We recall the map F from the preimage of $S_{\mathbb{S}^2}$ in $\mathbb{S}^4$ (which we denote Σ_S) and $\mathbb{S}^3$

$$F\left(0, \frac{\sqrt{2}}{2}\cos a, \frac{\sqrt{2}}{2}\sin a, \frac{\sqrt{2}}{2}\cos b, \frac{\sqrt{2}}{2}\sin b\right) = \left(0, e^{i(a+b)}\right) \tag{3.69}$$

We introduce the tubular neighbourhood $\mathcal{T}$ of Σ_S as $]-\epsilon, \epsilon[\times]-\epsilon, \epsilon[\times\Sigma$; we parametrize it as
$(\cos s, y) \times \left(0, \frac{\sqrt{2}}{2}e^{ia}, \frac{\sqrt{2}}{2}e^{ib}\right) \mapsto T_S\left(\cos s, y, \frac{\sqrt{2}}{2}e^{ia}, \frac{\sqrt{2}}{2}e^{ib}\right)$ with

$$T_S\left(\cos s, y, \frac{\sqrt{2}}{2}e^{ia}, \frac{\sqrt{2}}{2}e^{ib}\right) = \left(\cos s, \sin s\sqrt{1-y}\frac{\sqrt{2}}{2}e^{ia}, \sin s\sqrt{1+y}\frac{\sqrt{2}}{2}e^{ib}\right) \tag{3.70}$$

Thus the fibre of an element of Σ in the tubular neighbourhood $\mathcal{T}$ is parametrized by

$$(\cos s, y) \in]-\epsilon, +\epsilon[\times]-\epsilon, +\epsilon[.$$

We fix w close to $S_{\mathbb{S}^2}$ and we fix e^{ia}, e^{ib}: we look for $\cos s$, y close to 0 such that

$$\Phi_{kl}\left(T_S\left(\cos s, y, \frac{\sqrt{2}}{2}e^{ia}, \frac{\sqrt{2}}{2}e^{ib}\right)\right) = w \tag{3.71}$$

We compute $F\left(T_S\left(\cos s, y, \frac{\sqrt{2}}{2}e^{ia}, \frac{\sqrt{2}}{2}e^{ib}\right)\right)$

$$= \left(\cos\alpha(s), \sin\alpha(s) H\left(\sqrt{1-y}\frac{\sqrt{2}}{2}e^{ia}, \sin s\sqrt{1+y}\frac{\sqrt{2}}{2}e^{ib}\right)\right)$$

where $H : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is the Hopf fibration; thus we translate (3.71) into

$$\phi_{kl}\left(\cos\alpha(s), -y\sin\alpha(s), \sin\alpha(s)\sqrt{1-y^2}e^{i(a+b)}\right) = w \tag{3.72}$$

Similarly to (3.65) or rather its equivalent for $S_{\mathbb{S}^2}$, we rewrite (3.72) as

$$C(\cos\alpha(s) - iy\sin\alpha(s))^k(1 + o(\|(\cos s, y)\|)e^{il(a+b)} = w \tag{3.73}$$

This gives us k values for the couple $(\cos s, y)$.
To see that they are all of the same sign, we proceed as in Lemma 17 and we take the partial derivatives of (3.73) w.r.t. s and y. Using the fact that $\sin\alpha(s)$ is close to 1, we see that these partial derivatives are linearly independent. It follows that the fibres of the tubular neighbourhood $\mathcal{T}_S$ are always transverse to the preimages of points w close to S.
This concludes the proof of Lemma 14 1).

We now prove Lemma 14 2) using the definition above of singular multiple fibres. The preimage of $N_{\mathbb{S}^2}$ is

$$\Phi_{kl}^{-1}(N) = \{(\cos s, \sin s(\cos te^{ia}, \sin te^{ib})) \text{ with } \cos t\sin t = 0\} \tag{3.74}$$

A singular tubular neighbourhood will be given by

$$\mathcal{T}_N = \{(\cos s, \sin s(\cos te^{ia}, \sin te^{ib})) \text{ with } |\cos t\sin t| < \eta\} \tag{3.75}$$

for η small enough. Since $\cos^2 t + \sin^2 t = 1$, $\mathcal{T}_N$ will split into the union of $\mathcal{T}_1$ and $\mathcal{T}_2$ where

$$\mathcal{T}_1 \text{ (resp. } \mathcal{T}_2) = \{(\cos s, \sin s(\cos te^{ia}, \sin te^{ib})) \text{ with } |\cos t| < \eta \text{ (resp. } |\sin t| < \eta)\}. \tag{3.76}$$

Note that $\mathcal{T}_1$ and $\mathcal{T}_2$ intersect only at the two poles of $\mathbb{S}^4$.

Now fix $p \in \Phi_{kl}^{-1}(N_{\mathbb{S}^2})$.
If $p = (\cos s, \sin se^{ia}, 0)$ (resp. $p = (\cos s, 0, \sin se^{ib})$), then

$$\pi^{-1}(p) = \{(\cos s, \sin s\cos te^{ia}, \sin s\sin te^{i\theta})/|\sin t| < \eta\} \tag{3.77}$$

$$\text{(resp. } \pi^{-1}(p) = \{(\cos s, \sin s\cos te^{i\theta}, \sin s\sin te^{ib})/|\cos t| < \eta\}). \tag{3.78}$$

The fibre $\pi^{-1}(p)$ is parametrized by

$$z = \sin te^{i\theta} \text{ (resp. } z = \cos te^{i\theta}). \tag{3.79}$$

We now show that if w is close to $N_{\mathbb{S}^2}$ and p is of the form $p = (\cos s, \sin se^{ia}, 0)$, with $\cos s \neq \pm 1$, then $\pi^{-1}(p) \cap \Phi_{kl}^{-1}(w)$ contains k points and that these intersection points have the same sign. The proof of the same fact for p of the form $p = (\cos s, 0, \sin se^{ib})$ is identical.

We let $q \in \pi^{-1}(p)$ be a point of the type (3.77); we have

$$F(q) = (\cos\alpha(s), \sin\alpha(s)\cos 2t, \sin\alpha(s)\sin 2te^{i(\theta+a)}). \tag{3.80}$$

Changing variables in $\mathbb{S}^3$, we have

$$\sin ue^{i(a+\theta)} = \sin\alpha(s_0)\sin 2te^{i(\theta+a)} = 2\sin\alpha(s_0)\cos te^{ia}\sin te^{i\theta}$$

$$= 2\sin\alpha(s_0)\cos t e^{ia} z \tag{3.81}$$

where z is given by (3.79). It follows that $|\sin u| < 2\eta$. We also derive

$$\beta(u) = C(1 + o(t))[2\sin\alpha(s_0)\cos t]^l \sin^l t. \tag{3.82}$$

On the other hand,

$$\cos\alpha(s) + i\sin\alpha(s)\cos 2t = \cos u e^{i\psi}. \tag{3.83}$$

Since t is very small, we derive

$$\psi = \alpha(s_0) + o(t). \tag{3.84}$$

We can now write the restriction of $\Phi_{kl} = \varphi_{kl} \circ F(q)$ to $\pi^{-1}(p)$. To do this, we continue using the parameter z on $\pi^{-1}(p)$ (cf. (3.79)) and we identify a neighbourhood of $N_{\mathbb{S}^2}$ with a small disk D in $\mathbb{C}$. Using (3.82) we get

$$z = \sin t e^{i\theta} \mapsto C[2\sin\alpha(s)\cos t]^l \sin^l t(1 + o(|z|))e^{i[k\psi + l(a+\theta)]}. \tag{3.85}$$

In other words, there exists a complex number Z_0 (independent of s_0) such that we can rewrite (3.85) as

$$z \mapsto Z_0 \sin^l \alpha(s_0)(1 + o(|z|)z^l. \tag{3.86}$$

Hence, if w is a small enough non-zero complex number, more precisely, if

$$0 < |w| < \frac{1}{2}\eta^l |Z_0| \sin^l \alpha(s_0),$$

it has l preimages in $\pi^{-1}(p)$.

REMARK. We point out the contrast with the smooth multiple fibre case: the neighbourhood of $N_{\mathbb{S}^2}$ where we look for points with l preimages in $\pi^{-1}(X)$ depends on X and get smaller and smaller as X approaches the singularities of the singular fibre.

This being said, we proceed as in the smooth case to show that the l preimages have the same sign. The map given by (3.86) is a submersion and $\mathcal{T}_1 \setminus S_1$ and $\mathcal{T}_2 \setminus S_2$ are connected. Thus all the preimages in $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) have the same sign. Possibly after changing the orientation on one of the 2-spheres S_1 and S_2, we can ensure that these signs are all the same.

3.11 Appendix

To make this self contained we reproduce the computation of ([Bu]).
The metric $(g_{k,l})$ is expressed explicitly in terms of s, t, a, b by

$$g_{k,l} = \frac{2\left[ds^2 + \sin s^2 \left(dt^2 + \cos^2 t da^2 + \sin^2 t db^2\right)\right]}{\sqrt{\left(k^2 \sin^2 2t + l^2 \cos^2 2t\right)\left(\sin^4 s\right)/4 + l^2 \cos^2 s}}.$$

For the map F to be horizontally conformal of dilation λ, the function α must satisfy the following equation

$$\alpha^{'}(s)^2 = \frac{4 \sin^2 \alpha(s)}{\sin^2 s} \tag{3.87}$$

the equation (3.87) has an explicit solution, given by

$$\alpha(s) = 2 \arctan\left(\tan^2\left(\frac{s}{2}\right)\right).$$

The associated metric take the form :

$$\bar{g} = \frac{\sqrt{2}}{\left(k^2 \sin^2 s + l^2 \cos^2 s\right)^{1/4}} g_{\mathbb{S}^3}.$$

Chapter 4

Harmonic morphisms and Hermitian complex structures on $\mathbb{S}^2 \times \mathbb{S}^2$

In this chapter, we investigate the structure of a harmonic morphism F from $\mathbb{S}^2 \times \mathbb{S}^2$ to a 2-surface $\mathbb{S}^2$. Baird-Ou construct a family of harmonic morphism from an open set of $\mathbb{S}^2 \times \mathbb{S}^2$ into $\mathbb{S}^2$. We check that they are holomorphic with respect to one of the canonical complex structure.

4.1 Introduction

4.1.1 Background

Let M be a $2n$-dimensional real manifold. An almost complex structure J on M is a tensor field $J : TM \longrightarrow TM$ such that $J^2 = -I$.

Definition 14. *Let M be a $2n$-dimensional manifold. An almost complex structure J on M is called integrable if there exists an atlas $\{U_\alpha, \alpha : U_\alpha \longrightarrow \mathbb{R}^{2n}\}$ such that*

$$d\alpha \circ J = J_0 \circ d\alpha,$$

and the transition functions verify $d(\beta \circ \alpha^{-1})(z) \in GL(n; \mathbb{C})$. Here J_0 is the standard complex structure on $\mathbb{R}^{2n}$.

For any two vector fields X, Y on M, the Nijenhuis tensor N_J is defined as

$$NJ(X;Y) = [JX;JY] - J[JX;Y] - J[X;JY] - [X;Y].$$

One can prove that N_J is actually a tensor. We have the following theorem of Newlander-Nirenberg:

Theorem 11. *(Integrability theorem). An almost complex structure J is integrable if and only if the Nijenhuis tensor N_J vanishes.*

Now the vanishing of the Nijenhuis tensor can be viewed as a Frobenius integrability condition:

Lemma 18. *The set of the type (1, 0) vector fields is closed under the operation of Lie bracket if and only if $N_J \equiv 0$.*

Proof. Let X and Y be two real vector fields and we define the projections

$$P^{1,0}X = \frac{1}{2}(I - iJ)X; P^{0,1}Y = \frac{1}{2}(I + iJ)Y.$$

It is easy to show that

$$[P^{1,0}X; P^{1,0}Y] + iJ[P^{1,0}X; P^{1,0}Y] = -N_J(X,Y) - iJN_J(X,Y),$$

which is equivalent to

$$P^{0,1}([P^{1,0}X, P^{1,0}Y]) = P^{0,1}(-N_J(X,Y)).$$

Hence $N_J(X,Y)$ vanishes if and only if $[P^{1,0}X, P^{1,0}Y]$ is a $(1,0)$ vector. $\square$

In $\mathbb{C}^n$, we have the expression

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

or written in terms of J_0:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(1 - iJ_0)\frac{\partial}{\partial x_j}, \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(1 + iJ_0)\frac{\partial}{\partial x_j}.$$

Hence in the local coordinates z_j, the type (1,0) vector can be written as the combination $\sum_k a_k \frac{\partial}{\partial z_k}$ where a_k are complex valued functions on M. The Lie bracket of two (1,0) type vectors is still of the (1,0) type. If Σ is a 2-surface, then the Lie bracket of two (1,0) vectors is of type (1,0), hence we have

Theorem 12. *Every almost complex structure on a Riemann surface is integrable.*

Definition 15. *An integrable complex structure J on M is said to be hermitian if and only if for every p*

$$g(JX, JY) = g(X,Y) \text{ for all } X, Y \in T_pM.$$

4.1.2 The results

In this paper we investigate harmonic morphisms on open sets of $\mathbb{S}^2 \times \mathbb{S}^2$. J. Wood shows that the harmonic morphisms from an open subset of $\mathbb{C}P^1 \times \mathbb{C}P^1$ to a Riemann surfaces are holomorphic with respect to a Hermitian structure.
On $\mathbb{C}P^1$ we have two canonical complex structures $\pm J_0$ then on $\mathbb{C}P^1 \times \mathbb{C}P^1$ we obtain four canonical complex structures. Burns-Bartolomeis show that the only Hermitian structures on open subsets of $\mathbb{C}P^1 \times \mathbb{C}P^1$ are one of the four products of canonical complex structure $\pm J_0$.
On other hand Baird-Ou define harmonic morphism from open dense sets of $\mathbb{S}^2 \times \mathbb{S}^2$ into $\mathbb{S}^2$. We check that they are holomorphic w.r.t. one of these four structures.

Theorem 13. *([B-O]) There are two tori T_1^2 and T_2^2 in $\mathbb{S}^3$ such that there exists a family of harmonic morphisms $F : \mathbb{S}^3 \times \mathbb{S}^3 - (T_1^2 \cup T_2^2) \longrightarrow (\mathbb{S}^2, can)$ parametrized by quadruples of non-zero integers (k, l, m, n), given explicitly by the formula:*

$$F\left(\left(\cos s e^{ia}, \sin s e^{ib}\right), \left(\cos t e^{ic}, \sin t e^{id}\right)\right) = \left(\cos\alpha(s,t), \sin\alpha(s,t) e^{i(ka+lb+mc+nd)}\right),$$

where α is given by the solution of Equation

$$\frac{1}{\sin^2\alpha}\left(\left(\frac{\partial\alpha}{\partial s}\right)^2 + \left(\frac{\partial\alpha}{\partial t}\right)^2\right) = \frac{k^2}{\cos^2 s} + \frac{l^2}{\sin^2 s} + \frac{m^2}{\cos^2 t} + \frac{n^2}{\sin^2 t}. \tag{4.1}$$

For $|k| = |l|$ and $|m| = |n|$ the equation (4.1) is solved by

$$\alpha(s,t) = 2\tan^{-1}(A \tan^k s \tan^m t),$$

for a positive real constant A.

Corollary 3. *In the case $|k| = |l|$ and $|m| = |n|$, the harmonic morphism of the theorem factors to a harmonic morphism*

$$F : (\mathbb{S}^2 \times \mathbb{S}^2) \setminus \{two\ points\} \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3$$

Proposition 15. *Let $F : \mathbb{S}^2 \times \mathbb{S}^2 \setminus \{two\ points\} \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ a harmonic morphism defined by*

$$F\left(\left(\cos s, \sin s e^{iA}\right) \times \left(\cos t, \sin t e^{iC}\right)\right) = \left(\cos\alpha\left(\frac{s}{2}, \frac{t}{2}\right), \sin\alpha\left(\frac{s}{2}, \frac{t}{2}\right) e^{i(kA+mC)}\right),$$

with

$$\alpha\left(\frac{s}{2}, \frac{t}{2}\right) = 2\tan^{-1}\left(A \tan^k \frac{s}{2} \tan^m \frac{t}{2}\right).$$

Then F is holomorphic with respect to one of the four canonical complex structures on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Proof. To prove the theorem, we need to prove that for two orthogonal vertical vectors V_1 and V_2, we have $J_0(V_1) = \lambda V_2$ where J_0 is one of the four canonical

complex structures.
The tangent space $T(\mathbb{S}^2 \times \mathbb{S}^2)$ for $s \neq 0, \pi$ and $t \neq 0, \pi$ is generated by the vector fields $\frac{\partial}{\partial s}, \frac{\partial}{\partial A}, \frac{\partial}{\partial t}, \frac{\partial}{\partial C}$.
We have

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \left(-\sin\alpha\frac{\partial\alpha}{\partial s}, \cos\alpha\frac{\partial\alpha}{\partial s}e^{i(kA+mC)}\right), \\
\frac{\partial F}{\partial A} &= \left(0, \sin\alpha ike^{i(kA+mC)}\right), \\
\frac{\partial F}{\partial t} &= \left(-\sin\alpha\frac{\partial\alpha}{\partial t}, \cos\alpha\frac{\partial\alpha}{\partial t}e^{i(kA+mC)}\right), \\
\frac{\partial F}{\partial C} &= \left(0, im\sin\alpha e^{i(kA+mC)}\right).
\end{aligned}$$

So we need to calculate $\frac{\partial\alpha}{\partial s}$ and $\frac{\partial\alpha}{\partial t}$. By a small calculation we obtain the following

$$\begin{aligned}
\frac{\partial\alpha}{\partial s} &= \frac{A\tan^m\frac{t}{2}k(1+\tan^2\frac{s}{2})\tan^{(k-1)}\frac{t}{2}}{1+(A\tan^k\frac{s}{2}\tan^m\frac{t}{2})^2} \\
\frac{\partial\alpha}{\partial t} &= \frac{A\tan^k\frac{s}{2}m(1+\tan^2\frac{t}{2})\tan^{(m-1)}\frac{t}{2}}{1+(A\tan^k\frac{s}{2}\tan^m\frac{t}{2})^2} \\
\frac{\partial\alpha}{\partial t} &= \frac{\sin s}{\sin t}\frac{m}{k}\frac{\partial\alpha}{\partial s}. \qquad (4.2)
\end{aligned}$$

We can choose and easily check that the following vector fields are mutually orthogonal and vertical $(dF(V_1) = dF(V_2) = 0)$:

$$\begin{aligned}
V_1 &= m\frac{\partial}{\partial A} - k\frac{\partial}{\partial C} \\
V_2 &= -\frac{\partial\alpha}{\partial t}\frac{\partial}{\partial s} + \frac{\partial\alpha}{\partial s}\frac{\partial}{\partial t},
\end{aligned}$$

Using (4.2) we find that

$$\begin{aligned}
V_2 &= \frac{\partial\alpha}{\partial s}\left[-\frac{\sin s}{\sin t}\frac{m}{k}\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right] \\
&= k^{-1}\sin^{-1}t\frac{\partial\alpha}{\partial s}\left[-\sin sm\frac{\partial}{\partial s} + \sin tk\frac{\partial}{\partial t}\right].
\end{aligned}$$

We let J_0 be one the four canonical structures defined by

$$\begin{aligned}
J_0(\frac{\partial}{\partial s}) &= \frac{1}{\sin s}\frac{\partial}{\partial A} \\
J_0(\frac{\partial}{\partial t}) &= \frac{1}{\sin t}\frac{\partial}{\partial C} \\
J_0(\frac{\partial}{\partial A}) &= -\sin s\frac{\partial}{\partial s} \\
J(_0\frac{\partial}{\partial C}) &= -\sin t\frac{\partial}{\partial t}.
\end{aligned}$$

Back to V_1 and V_2, appliying J_0 on V_1 then for some constant $\lambda(k, m)$

$$\begin{aligned} J_0(V_1) &= mJ_0(\frac{\partial}{\partial A}) - kJ_0(\frac{\partial}{\partial C}) \\ &= -m \sin s \frac{\partial}{\partial s} + k \sin t \frac{\partial}{\partial t} \\ &= \lambda(k, m) V_2. \end{aligned}$$

□

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